A STUDY OF CATEGORIES OVER AND BELOW OBJECTS

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PREFACE

This thesis entitled '' A Study of Categories Over and Below Objects '' is the research work done by me since 29th of November,1970 under the encouraging, inspiring and kind supervision of Professor M. A. Kazim, Department of Mathematics and Statistics, Aligarh Muslim University, Aligarh. The work was all along financially supported by the Council of Scientific and Industrial Research , India.

The concept of category over an object and below an object was firstly introduced by A. Grothendieck and I. Dieudonné in their paper '' Eléments de géometric algébrique'' Vol. III, Pub. Math. Inst. des Haustes Etudes, 11 (1961), 1-167. They call $\mathcal{C}/A$ as the category of objects above A. Dually $A/\mathcal{C}$ can be defined. Later on Maclane [12] , in 1965 gave the same definitions of these categories. Bucur [3] defined $\mathcal{C}/S$ as the category , whose objects are all pairs (A,α) , where A is an object of the category $\mathcal{C}$ and α is a morphism from A to S in $\mathcal{C}$, and the class of whose morphisms consists of all morphisms $f : A \longrightarrow B$ in $\mathcal{C}$, $\forall$ A , B $\in \mathcal{C}$, such that the following diagrams

$$\begin{tikzcd} A \arrow[dr] \arrow[dd] & \\ & S \\ B \arrow[ur] & \end{tikzcd}$$

commute. He added to the above definition the concept of fibered products and coproducts over a particular scheme and obtained some properties of fibered products.

The purpose of the present work is to study thoroughly related to $\mathcal{C}$, the categories over an object and below an object, denoted by us as $\mathcal{C}_S$ and $\mathcal{C}^S$ respectively. We investigated their structural properties and those, which are preserved both ways in relation to $\mathcal{C}$ directly or under certain conditions, like: products, coproducts (Sec. 1.2) equalizers, coequalizers (Sec. 1.3), pullbacks, pushouts (Sec. 1.4), intersections, cointersections (Sec. 1.5), completeness, cocompleteness (Sec. 1.6), filteredness, cofilteredness (Sec. 1.7) and Abelianness (1.9). We further introduced new concepts , which extend the relative study of $\mathcal{C}_S$, $\mathcal{C}^S$ and $\mathcal{C}$, as fibered projectives (Sec. 3.1), fibered injectives (Sec. 3.2), fibered injective hulls and projective covers (Sec. 3.4), fibered generators (Sec. 3.5), fibered reflectivity (Sec. 3.6), some fibered morphisms (regular (Sec. 4), strong (Sec. 4.3), extremal (Sec. 4.5)) and some fibered factorizations (regular (Sec. 4.2), canonical (Sec. 4.4) extremal epi-mono (Sec. 4.6)), a special type of category called V-category (Sec. 1.10), special type of functors F/F', G/G' on $\mathcal{C}_S$ / $\mathcal{C}^S$ (Sec. 2.1), T_f , T^f (Sec. 2.2), T_{cat} , T^{cat} : $\mathcal{C}$ ----> $\mathcal{C}at$ (Sec. 2.3) and T_S : $\mathcal{C}_S$ ----> $\mathcal{C}_{S'}$ induced by a functor T : $\mathcal{C}$ ----> $\mathcal{C}'$ (Sec. 2.4).

We have also generalized the concept of these categories in different ways : categories over and below for two objects (Sec. 5.1), for a countable number of objects (Sec. 5.2) , categories which are both over and below objects at the same-time (Sec. 5.3) and categories over categories like ${}^{\mathcal{A}}_{\mathcal{B}}S$, $S^{\mathcal{A}}_{\mathcal{B}}$, ${}^{\mathcal{A}}_{\mathcal{B}}\mathcal{C}$, $\mathcal{C}^{\mathcal{A}}_{\mathcal{B}}$, ${}^{\mathcal{A}_{\mathcal{B}}}S$, $S^{\mathcal{B}^{\mathcal{A}}}$, ${}^{\mathcal{A}_{\mathcal{B}}}\mathcal{C}$ and $\mathcal{C}^{\mathcal{B}^{\mathcal{A}}}$, where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are arbitrary categories and S is an object of $\mathcal{C}$ (Sec.5.4) .

Apart from this, we have also studied some properties of characteristic subobjects and certain types of functors and factorizations.

The thesis consists of six chapters containing 46 theorems, 131 propositions, 29 lemmas , 7 corollaries , 8 remarks in all. The zero ''0'' chapter provides the preliminaries, which are used in the later chapters. Apart from important notions , some results have also been quoted without proof for further use.

Chapter I gives the structural study of $\mathcal{C}_S$ and $\mathcal{C}^S$ with respect to $\mathcal{C}$. We actually find out that, if $\mathcal{C}$ has initial objects, coproducts, pullbacks, pushouts, equalizers, coequalizers, intersections and cointersections, then $\mathcal{C}_S$ also has initial objects, coproducts, pullbacks, pushouts, equalizers, coequalizers, intersections and cointersections respectively. The converse is true only when we put the restriction that S is a terminal object in $\mathcal{C}$. We, therefore, conclude : If S is a terminal object in $\mathcal{C}$, then (i) $\mathcal{C}$ is right complete if and only if $\mathcal{C}_S$ is right complete, (ii) $\mathcal{C}$ is filtered if and only if $\mathcal{C}_S$ is filtered, (iii) $\mathcal{C}$ is abelian iff $\mathcal{C}_S$ is abelian. Similar results hold dually for $\mathcal{C}^S$. In the end, we find that if $\mathcal{C}$ is a V-category, the restriction of S being terminal is removed in some converse cases like : pullbacks, intersections and normality.

Chapter II is mostly devoted to the study of special types of functors introduced on $\mathcal{C}_S$. We first show that if S is terminal object then $\mathcal{C} \approx \mathcal{C}_S$ by using functors

F and G. We have proved the functor $T_f : \mathcal{C}_S \longrightarrow \mathcal{C}_{S'}$ is additive if $\mathcal{C}_S$ is an additive category. $T_{Cat} : \mathcal{C} \longrightarrow \mathcal{C}at$ is additive (Lemma 2.3) and left limit preserving (Proposition 2.3) , $T_S : \mathcal{C}_S \longrightarrow \mathcal{C}'_{S'}$, where $S' = T(S)$, is faithful, full, exact, embedding limit preserving if $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is also faithful, faithful full , exact, embedding and limit preserving and has left adjoint if T has. We find a commutativity-like relation between T_f and T_S . In the last section, we introduce $\hat{T}_S : \mathcal{C}_S \longrightarrow \mathcal{C}'_{S'}$, where S' is a terminal object of $\mathcal{C}'$ and obtained some important properties out of which one is the existence of natural transformation for two functors of this type.

In Chapter III, fibered structures have been introduced. Some of the main investigations, out of 41 propositions and 22 theorems established in this chapter, are (1) An object P is projective (i) iff it is fibered projective over a terminal object, (ii) if and only if it is cofibered projective over an initial object, (iii) and if P belongs to a V-category, in which every epimorphism in $\mathcal{C}_S$ is epi in $\mathcal{C}$, it is fibered projective over any object. (2) (i) If $S = \prod S_i$, then P is fibered projective over S iff S is fibered projective over each S_i, (Theorem 3.3) (ii) if $P = \oplus P_i$, then P is fibered projective over S iff each P_i is fibered projective over S (Theorem 3.5). (3) In V-categories with coequalizers, an object is generator iff it is fibered generator over any object, (ii) In V-categories with coproduct , if $\{U_i\}_{i \in I}$ is a family of fibered generators over S, then $\oplus U_i$ is fibered generator over S (Proposition 3.29). In the end of this chapter, a relative study of reflective and fibered reflective subcategories over an object of the category has been made. One of the important result is : If $\mathcal{C}'$ is reflective subcategory of a V-category $\mathcal{C}$, then $\mathcal{C}'$ is fibered reflective subcategory of $\mathcal{C}$ over all objects of $\mathcal{C}'$.

Chapter IV deals with the fibration of the notions of extremal mono and epi by Isbell [6], of regular and strong epi and mono, and their regular and canonical factorisations by Kelly [10] , of extremal mono-epi factorisations of Herrilich [8] . A relative study has been made between these concepts and the corresponding fibered concepts introduced here over an object of a category. Some of our important investigations are as : In V-category , a regular epi/mono is fibered/cofibered regular epi/mono over an object S of the category. Also, if a morphism has a regular epi-factorisation then it has fibered regular epi-factorisation. The converse problem holds only if S is a terminal object. The behaviour of other fibered factorisations is nearly similar.

In Chapter V , we generalise the notion of categories over an object and below an object in several respects. Categories like $\mathcal{C}_{S_1,S_2}$, $\mathcal{C}^{S_1,S_2}$, $\mathcal{C}_{S_1,S_2,\ldots,S_n,\ldots}$, $\mathcal{C}^{S_1,S_2,\ldots,S_n,\ldots}$ have been defined and their study is similar to our study of categories over a single object, which can be achieved by repeated applications of previous results. One very interesting phenomenon which we have shown is that $\mathcal{C}_{S_1,S_2,\ldots,S_n,\ldots} \approx \mathcal{C}_{\pi S_i}$ and any $\mathcal{C}_{S_1,\ldots,S_n,\ldots}$ can be embedded in $\mathcal{C}_{S_i}$, for any i . We close this chapter by giving some new constructions of categories over and below categories.

Lastly, we give an Appendix which contains the copies of two papers prepared during this period on topics not connected with the title of our thesis. One is entitled '' A Note On Characteristic Subobjects'' and the other '' On Certain Types of Functors and Factorisations''. In connection with the latter , it will be in right fitness to thank Professor G. M. Kelly of University of Sydney for his kind help in providing the reprints of his papers some of which have not only been used in this paper but also in the thesis.

To meet the requirement of Clause No. VIII of Chapter XXV Academic Ordinances, apart from the short resume in the preface of the thesis, every chapter and its sections are generally well-equipped with comprehensive introductions pointing out the main theorems proved and concepts introduced followed by a short list of relevant references at its end.

Finally, I again express my sense of gratitude to my supervisor Professor M. A. Kasim for his careful and continued guidance during the preparation of this thesis.

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SYMBOLS USED

The following is the list of symbols and abbreviations unless or otherwise they are mentioned :

$\mathcal{C}$	: Category
$A, B, .$	: Objects of a category
$A \in \mathcal{C}$	: A is an object of $\mathcal{C}$
$\alpha, \beta, \ldots, f, g ..$	: Morphisms of a category
$\alpha \in \mathcal{C}$	: α is a morphism in $\mathcal{C}$ or of $\mathcal{C}$
$[A,B]_{\mathcal{C}}$ or $[A,B]$	: Set of all morphisms from A to B in $\mathcal{C}$
$\mathrm{Hom}_{\mathcal{C}}(A,B)$	: Set of all morphisms from A to B in $\mathcal{C}$
$\mathcal{C}^*$, $\mathcal{C}^o$	: Dual category of $\mathcal{C}$
$\prod_{i \in I} A_i$	: Product of objects $\{A_i\}_{i \in I}$
$\bigoplus_{i \in I} A_i$	: Coproduct (Sum) of objects $\{A_i\}_{i \in I}$
Equa (α, β)	: Equaliser of α and β
Coequa (α, β)	: Coequaliser of α and β
$\mathcal{C}_S$	: Category over an object S
$\mathcal{C}^S$	: Category below an object S
(A, α)	: Object of $\mathcal{C}_S$
(α, A)	: Object of $\mathcal{C}^S$
$m : (A, \alpha) \longrightarrow (B, \beta)$	: a morphism in $\mathcal{C}_S$

Ens	:	Category of sets
Gr	:	Category of all groups
$\mathcal{M}_R$	:	Category of all Right R-Modules
$\mathcal{T}$	:	Category of topological spaces
0	:	Zero object
P	:	Projective object
Q	:	Injective object
F.G.S.T.	:	Functors
$\eta : T_1 \longrightarrow T_2$	:	A natural transformation from T_1 to T_2
$\alpha : T \dashv S$	:	α is a natural equivalence of adjoint of T to S
$T : \mathcal{C}_1 \approx \mathcal{C}_2$	:	T is a natural equivalence between $\mathcal{C}_1$ and $\mathcal{C}_2$
$\approx$	:	Equivalence
$T_1 \circ T_2$	:	T_1 composition T_2
mono	:	monomorphisms
epi	:	epimorphisms
Pro	:	Proposition
def	:	definition
Theo	:	Theorem
Sec	:	Section
$\longrightarrow$	:	imply
$\exists$	:	there exist
$\equiv$	:	same

$\cup$	:	Union
$\cap$	:	Intersection
$\subseteq$	:	Contain in
$\leq$	:	Substructure
$f\|A$	:	f is restricted to A
$A/B \ldots C/D$	:	A corresponds to C, B corresponds to D
$\forall$	:	for all .

CHAPTER 0

PRELIMINARIES

In this chapter, definitions, general concepts, notations and results, which have been taken granted in the latter chapters of this thesis, are provided. Generally, the definitions and notations are taken from [1], [3] and [13] and from research papers [2], [12] and [16]. The proof of these results are not given but a proper reference has been noted.

0.1. Notions in the theory of categories

In this section, we introduce the definitions and general concepts involved in the category structure such as products, coproducts, pullbacks, pushouts, equalizers, coequalizers, intersections, cointersections, kernels, cokernels, normality, exactness, additivity and abelianness. We give some results, which have been used in the work ahead.

0.1.1. Definitions and examples

Definition 0.1. Category (An objective approach)

A category $\mathcal{C}$ is a system consists of a class $\mathcal{O}$ of objects $\{A , B, \ldots \}$ together with a class $\mathcal{M}$ of morphisms $\{\alpha , \beta, \ldots, f , g, \ldots \}$, which is a disjoint union of the form

$$\mathcal{M} = \bigcup_{(A , B) \in \mathcal{O}\times\mathcal{O}} [A,B]_{\mathcal{C}}$$

satisfying.

For each triple (A,B,C) of members of $\mathcal{C}$, we are to have a function from $[B,C]_{\mathcal{C}} \times [A,B]_{\mathcal{C}}$ to $[A,C]_{\mathcal{C}}$. The image of the pair (β,α) under this function will be called the composition of β by α , and will be denoted by $\beta\alpha$. The composition functions are subject to two axioms :

(a) <u>Associativity</u> : Whenever the compositions make sense we have

$$(\gamma\beta)\alpha = \gamma(\beta\alpha).$$

(b) <u>Existense of identity</u> : For each object $A \in \mathcal{O}$, we have an element $I_A \in [A,A]$, called the identity morphism from A to A , such that $I_A\alpha = \alpha$ and $\beta I_A = \beta$ whenever the compositions make sense. ([13] , p.1).

Remark 0.1. To avoid logical difficulties, we postulate that each $[A,B]_{\mathcal{C}}$ is a set (possibly void). When there is danger of no confusion we shall write $[A,B]$ inplace of $[A,B]_{\mathcal{C}}$. Also a morphism $\alpha \in [A,B]_{\mathcal{C}}$, we shall write as $\alpha \in \mathcal{C}$.

Remark 0.2. An object $A \in \mathcal{O}$ for a category $\mathcal{C}$, we shall write $A \in \mathcal{C}$.

Remark 0.3. The condition 'b' on morphisms asserts that the rule $A \longrightarrow I_A$ provides a one to one correspondence between the class of all objects and the class of all identity morphisms. Thus a category can be represented as $= \{ \mathcal{O}, \mathcal{M}, 1_{\mathcal{M}} \}$.

Definition 0.2. <u>Category (Non-objective approach)</u>

A category $\mathcal{C}$ can also be defined as a class. $\mathcal{M}$, together with a partially defined binary operation on $\mathcal{M}$, called composition. The image of the pair (β,α) under this operation is $\beta\alpha$ (if defined) and satisfies ;

(i) If either $(\gamma\beta)\alpha$ or $\gamma(\beta\alpha)$ is defined, then the other is defined , and they are equal.

(ii) If $\gamma\beta$ and $\beta\alpha$ are defined and β is an identity, then $\gamma\alpha$ is defined.

(iii) Given $\alpha \in \mathcal{M}$ there are identities I_L and I_R in $\mathcal{C}$ such that $I_L\alpha$ and αI_R are defined and equal to α.

(iv) For any pair of identities I_L and I_R, the class $\{\alpha \in \mathcal{M} \mid (I_L\alpha)I_R \text{ is defined}\}$ is a set ([13] p.2).

Remark 0.4. Definitions 0.1 and 0.2 are equivalent ([13], p.2).

Category $\mathcal{C}$ is <u>small</u> if class of objects is a set.

Definition 0.3. A category $\mathcal{C}' = \{\mathcal{O}', \mathcal{M}', i_{\mathcal{M}}\}$ is a <u>subcategory</u> of the category $\mathcal{C} = \{\mathcal{O}, \mathcal{M}. i_{\mathcal{M}}\}$ if and only if

(i) $\mathcal{O}' \subseteq \mathcal{O}$.

(ii) $[A,B]_{\mathcal{C}'} \subseteq [A,B]_{\mathcal{C}}$ for all pairs $(A,B) \in \mathcal{O}' \times \mathcal{O}'$.

(iii) The composition of any two morphisms in $\mathcal{C}'$ is the same as their composition in $\mathcal{C}$.

(iv) I_A is the same in $\mathcal{C}'$ as in $\mathcal{C}$ for all $A \in \mathcal{C}'$.

Definition 0.4. The <u>dual category</u>, denoted by $\mathcal{C}^* = \{\mathcal{O}^*, \mathcal{M}^*, i_{\mathcal{M}^*}$ of a category $\mathcal{C} = \{\mathcal{O}, \mathcal{M}, i_{\mathcal{M}}\}$ is such that

(i) $\mathcal{O}^* = \mathcal{O}$

(ii) $[A,B]_{\mathcal{C}^*} = [B,A]_{\mathcal{C}}$ and

(iii) the composition of $\alpha \in [A,B]_{\mathcal{C}^*}$ and $\beta \in [B,C]_{\mathcal{C}^*}$ denoted as $\alpha\beta$ in $\mathcal{C}^* = \beta\alpha$ in $\mathcal{C}$.

Remark 0.5 (i) $(\mathcal{C}^*)^* = \mathcal{C}$ ([13], p.4)

(ii) $\mathcal{C}^*$ is some times denoted as $\mathcal{C}^0$.

Definition 0.5. An object T is a <u>terminal object</u> in a category $\mathcal{C}$ if to each object A in $\mathcal{C}$, there is exactly one morphism $A \longrightarrow T$. In other words, if $[A,T]$ = singleton, for all objects A in $\mathcal{C}$.

Definition 0.6. An object I in $\mathcal{C}$ is called an <u>initial object</u> if $[I,A]$ = singleton for all objects $A \in \mathcal{C}$.

The following is the list of terminal and initial objects in certain categories [12] :

Example No.	Category	Notation	Morphisms	Initial objects	Terminal objects
0.1	Category of sets	Ens	Functions	Empty set	All singletons
0.2	Category of all groups	Gr	Group homo-morphisms	Identity group: e	e
0.3	Category of all abelian groups	$\mathcal{A}b$	Group homo-morphisms	e	e

Example No.	Category	Notation	Morphisms	Initial objects	Terminal objects
0.4	Category of all R-modules	$\mathcal{M}_R$	R-homomorphisms	Trivial R-module: 0	0
0.5	Category of all topological spaces	$\mathcal{T}op$	Continuous functions	Empty space	One point spaces
0.6	Category of sets with base points	Ens^*	Base point preserving functions	All singletons with base points	All singletons with base points
0.7	Category of topological space with base points	$\mathcal{T}op^*$	Base point preserving continuous functions	The spaces of base point only	One point space where the point is base point.

Definition 0.7. An object Z of a category $\mathcal{C}$ is called a <u>zero object</u> if Z is both initial and terminal object ([13],p.14). We denoted if by '0'.

Example 0.8. Categories Gr, $\mathcal{A}b$ and $\mathcal{M}_R$ have zero objects.

0.1.2. <u>Special morphisms</u>

Definition 0.8. A morphism A ---> B is a <u>zero morphism</u> if it factors through 0 (zero object) viz, A ---> B = A ---> 0---> B.

Definition 0.9. A category $\mathcal{C}$ is called a <u>category with zero object</u> if it contains a zero object.

Definition 0.10. A morphism $m : A \longrightarrow B$ is called a <u>monomorphism</u> if and only if for each pair of morphisms $f , g : C \longrightarrow A$ such that

$$C \xrightarrow{f} A \xrightarrow{m} B = C \xrightarrow{g} A \xrightarrow{m} B$$

implies $f = g$, [3] . In other words iff m is left cancellable ([13] , p.17).

Definition 0.10*. Dually , a morphism $e : A \longrightarrow B$ is called an <u>epimorphism</u> if and only if it is right cancellable.

Definition 0.11. A morphism $\alpha : A \longrightarrow B$ is called an isomorphism if and only if there exists a morphism $\beta : B \longrightarrow A$ such that $\alpha\beta = I_B$ and $\beta\alpha = I_A$.

Remark 0.6. (i) Every monomorphism in a category need not be one to one.

<u>Example 0.9</u>. Consider the category of pathwise connected topological spaces with base points and continuous maps (taking base points into base points). Then if A is a covering space over B , the covering map is continuous , but not in general one-one , for example , S^n over P^n. However, since if the compositions of any two morphisms into A with the covering map are equal , the morphisms are equal , the

covering map is a monomorphism ([4], p.156).

(ii) Every epimorphism in a category need not be onto.

Example 0.10. Let X be a set and τ_1, τ_2 be two topologies on X such that $\tau_2 < \tau_1$. Then morphism f: $\tau_1 \longrightarrow \tau_2$ is an epimorphism in the category of topological spaces but not onto.

Example 0.11. Let $\mathcal{C}$ be the category whose objects are Abelian Topological groups (seperated in the sense of Hausdorff) whose morphisms are continuous homomorphisms. Then the morphism e : A $\longrightarrow$ B is epimorphism if and only if the closur of e(A) viz $\overline{e(A)}$ = B. So if e(A) may not be equal to B $\Longrightarrow$ e is not onto even epimorphism ([1] , p.5).

(iii) A mprphism , which is both mono and epi may not be an isomorphism.

Example 0.12. Consider the following objects and morphism in the category of Abelian Hausdorff topological groups :

R is the additive topological group of real numbers, Q is the additive topological group of rational numbers , u : Q $\longrightarrow$ R is the inclusion of Q into R and also onto but not an isomorphism ([2] , p.6).

Definition 0.10. A morphism $\alpha : A \longrightarrow B$ is called a <u>retraction</u> if there exists a morphism $\beta : B \longrightarrow A$ such that $\alpha\beta = I_B$. Dually α is called a <u>coretraction</u> if there exists $\beta' : B \longrightarrow A$ such that $\beta'\alpha = I_A$.

If α is $A \longrightarrow B$ is retraction / coretraction, then A is called coretract/retract of B.

Remark 0.7. A morphism which is both retraction and coretraction is an isomorphism ([13] , p.5).

Proposition 0.1. If $\alpha : A \longrightarrow B$ is a coretraction and is also an epimorphism , then it is an isomorphism ([13] ,p.6).

Proposition 0.2. If $\alpha : B \longrightarrow A$ is a retraction and is also a monomorphism , then it is an isomorphism ([13] ,p.6).

Definition 0.11. If $\alpha : A' \longrightarrow A$ is a monomorphism , then we say that A' is a <u>subobject</u> of A , and we refer to α as the inclusion of A' in A. Dually, if $p : A \longrightarrow A''$ is an epimorphism then we say that A'' is a <u>quotient object</u> of A. ([13] , p.6)

Remark 0.8. The above definition is taken from Mitchell [13] but Freyd [3] gives this definition in the following way :

Definition 0.12. A <u>subobject</u> of an object A is an <u>equivalence class of monomorphisms</u> into B. Dually, a quotient object of A is equivalence class of epimorphism from A.

Two monomorphisms $A_1 \xrightarrow{m_1} A$ and $A_2 \xrightarrow{m_2} A$ are said to be equivalent if there are morphisms $A_1 \longrightarrow A_2$ and $A_2 \longrightarrow A_1$ such that the following triangles

$$\begin{tikzcd} A_1 \arrow[d] \arrow[dr, "m_1"] & \\ A_2 \arrow[r, "m_2"] & A \end{tikzcd} \qquad \begin{tikzcd} A_2 \arrow[d] \arrow[dr, "m_2"] & \\ A_1 \arrow[r, "m_1"] & A \end{tikzcd}$$

are commutative ([3] , p. 19).

Remark 0.9. (i) The relation '' is a subobject of'' is transitive in the sense of Definition 0.11 , because composition of two monomorphisms is a monomorphism. But this relatiàn is not transitive in the sense of Definition 0.12. Indeed , subobjects , as defined in Definition 0.12 , do not have subobjects.

0.1.3. <u>Products and coproducts</u>

Definition 0.13. Let $\{A_i\}_{i \in I}$ be a family of objects in a category $\mathcal{C}$. A product for the family is a family of morphisms $\{p_i : A \longrightarrow A_i\}_{i \in I}$ such that for any other family of morphisms $\{p'_i : A' \longrightarrow A_i\}_{i \in I}$ there exists a unique morphisms $\gamma : A' \longrightarrow A$ such that the following diagram

$$\begin{tikzcd} A' \arrow[d, "\gamma"'] \arrow[dr, "p'_i"] & \\ A \arrow[r, "p_i"'] & A_i \end{tikzcd}$$

is commutative , for all $i \in I$.

The morphisms p_i's are called canonical projections.

Remark 0.10. The product is unique upto isomorphism ([3] ,p.23). Some times this unique object A is denoted by $\prod_{i \in I} A_i$.

Dually , we have the following definition :

Definition 0.14. A coproduct (Sum) of a family $\{A_i\}_{i \in I}$ is a family of morphism $\{A_i \xrightarrow{u_i} S\}_{i \in I}$ such that for any other family of morphisms $\{A_i \longrightarrow S'\}_{i \in I}$ there exists a unique morphism $\delta : S \longrightarrow S'$ such that the following diagram

$$\begin{tikzcd}
 & S \arrow[dd, "s"] \\
A_i \arrow[ur, "u_i"] \arrow[dr, "u_i'"'] & \\
 & S'
\end{tikzcd}$$

is commutative for all $i \in I$.

The morphisms $u_i : A_i \longrightarrow S$, are called <u>canonical injections</u>.

Remark 0.11. Coproduct is unique upto isomorphism ([3],[2],p.25)
This unique sum S is denoted by $\bigoplus_{i \in I} A_i$ also.

In the well known categories , the word ' coproduct' means:

Categories	Sum
Ens	Disjoint union
Abelian groups ($\mathcal{A}b$)	Direct sum (cartesion product)
All groups (Gr)	Free product
Commutative Rings (Comm R_g)	Tensor products
Modules over R ($\mathcal{M}_R$)	Direct sum.

Definition 0.15. A category is said to be a <u>category with products/ coproducts</u> if every family of objects in the category has a product/coproduct in the category.

Remark 0.12. If the index set I is finite we say product is finite product and category is called <u>category with finite products</u> .

0.1.4. <u>Equalizers and coequalizers</u>

Definition 0.16. A monomorphism $u : K \longrightarrow A$ is called the <u>equalizer</u> or difference Kernel of two morphisms α , $\beta : A \longrightarrow B$ if

(i) $\alpha u = \beta u$, and

(ii) for any other morphism $u' : K' \longrightarrow A$ such that $\alpha u' = \beta u'$ there exists a unique morphism $\gamma : K' \longrightarrow K$ such that the following triangle

$$\begin{tikzcd} K' \arrow[dr] \arrow[dd] & \\ & A \\ K \arrow[ur] & \end{tikzcd}$$

is commutative.

Definition 0.15*. Dually , an epimorphism $B \xrightarrow{p} C$ is called <u>coequalizer</u> or difference cokernel of two morphisms α , $\beta : A \longrightarrow B$ if

(i) $p\alpha = p\beta$, and

(ii) for any other morphism $p' : B \longrightarrow C'$ such that $p\alpha = p'\beta$, there exists a unique morphism $\delta : C \longrightarrow C'$ such that the following diagram

$$\begin{tikzcd} & C \arrow[dd, "\delta"] \\ B \arrow[ur, "p"] \arrow[dr, "p'"'] & \\ & C' \end{tikzcd}$$

is commutative ([3] , p.21 , [13] , p.7).

Remark 0.13. We denote equalizer/coequalizer of α and β by equa/coequa (α,β).

Definition 0.17. We say that a <u>category $\mathcal{C}$ has equalizers/ coequalizers</u> if every pair of morphisms α , $\beta : A \longrightarrow B$ for all A , $B \in \mathcal{C}$ has an equalizer/ a coequalizer in $\mathcal{C}$.

0.1.5. <u>Pullbacks and Pushouts</u>

Definition 0.18. For two morphisms $\alpha_1 : A_1 \longrightarrow A$ and $\alpha_2 : A_2 \longrightarrow A$, with a common codomain , the following commutative square

$$\begin{tikzcd} P \arrow[r, "\beta_2"] \arrow[d, "\beta_1"'] & A_2 \arrow[d, "\alpha_2"] \\ A_1 \arrow[r, "\alpha_1"] & A \end{tikzcd}$$

is called a pullback square for α_1 and α_2 if for every pair of morphisms $\beta_1' : P' \longrightarrow A_1$ and $\beta_2' : P' \longrightarrow A_2$ such that $\alpha_1\beta_1' = \alpha_2\beta_2'$, there exists a unique morphism $\gamma : P' \longrightarrow P$ such that $\beta_1' = \beta_1\gamma$ and $\beta_2' = \beta_2\gamma$.

Definition 0.18*. Dually, we define <u>pushout diagram</u> for a pair of morphisms $\alpha_1 : A \longrightarrow A_1$ and $\alpha_2 : A \longrightarrow A_2$, with common domain, is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"'] & A_1 \arrow[d, "\beta_1"] \\ A_2 \arrow[r, "\beta_2"] & Q \end{tikzcd}$$

such that for every pair of morphisms $\beta_1' : A_1 \longrightarrow Q'$ and $\beta_2' : A_2 \longrightarrow Q'$ such that $\beta_1'\alpha_1 = \beta_2'\alpha_2$, there exists a unique morphism $\gamma : Q \longrightarrow Q'$ such that $\gamma\beta_1 = \beta_1'$, $\gamma\beta_2 = \beta_2'$.

Definition 0.19. A <u>category is said to have pullbacks/pushouts</u> if every pair of morphisms $\alpha_i : A_i \longrightarrow A \mid \alpha_i : A \longrightarrow A_i$, $i = 1,2$, has a pullback / pushout in $\mathcal{C}$.

Zaidi, S.M.A. [15] generalized the definition 0.18 as follows

Definition 0.20. Let $\{\alpha_i : A_i \longrightarrow A \mid i = 1,2,\ldots n,\ n \geq 2\}$ be a family of morphisms in a category $\mathcal{C}$, an object $P \in \mathcal{C}$ together with a family of morphisms $\{\beta_i : P \longrightarrow A_i \mid i=1,2,\ldots n\}$ is called the generalised pullback of the given family of morphisms if the following axioms hold :

(i) $\alpha_1\beta_1 = \alpha_2\beta_2 = \ldots = \alpha_n\beta_n$.

(ii) For any object $P' \in \mathcal{C}$ with a family of morphisms $\{\beta_i' : P' \longrightarrow A_i \mid i = 1 \ldots n\}$ such that

$$\alpha_1\beta_1' = \ldots\ldots = \alpha_n\beta_n' ,$$

there exists a unique morphis $\gamma : P' \longrightarrow P$ such that

$$\beta_i\gamma = \beta_i' , \quad \text{for all } i = 1,2,\ldots,n.$$

Dually, generalised pushout is defined as :

Let $\{\alpha_i : A \longrightarrow A_i \mid i = 1,2,\ldots, n \geq 2\}$ be a family of morphisms in a category $\mathcal{C}$. Then an object $Q \in \mathcal{C}$ together with a family of morphisms $\{\beta_i : A_i \longrightarrow Q \mid i = 1,2,\ldots,n\}$ is called the generalised pushout of the given family of morphisms if the following axioms hold :

(i) $\beta_1\alpha_1 = \beta_2\alpha_2 = \cdots = \beta_n\alpha_n$.

(ii) For any object $Q' \in \mathcal{C}$ with a family of morphisms

$$\{\beta_i' : A_i \longrightarrow Q' \mid i = 1,2,\ldots,n\} \text{ such that}$$

$$\beta_1' \alpha_1 = \beta_2' \alpha_2 = \cdots = \beta_n' \alpha_n ,$$

there exists a unique morphism $\gamma : Q \longrightarrow Q'$ such that

$$\gamma\beta_i = \beta_i' \quad \text{for all } i = 1,2,\ldots,n.$$

Remark 0.14. For $n = 2$, the concepts reduces to simple pullbacks and simple pushouts.

Theorem 0.1. A category has generalised pullbacks/pushouts if and only if it has simple pullbacks/pushouts [16] .

0.6 Intersections and cointersections

Definition 0. . Let $\{u_i : A_i \longrightarrow A\}$ be a family of subobjects of A. Then a morphism $u : B \longrightarrow A$ is called the intersection of the family of $u = u_i v_i$, for each i , where $v_i : B \longrightarrow A_i$, and for every other morphism $u' : C \longrightarrow A$ satisfying $u' = u_i v_i'$, $\forall i$, for some $v_i' : C \longrightarrow A_i$, there exists a unique morphism $\gamma : C \longrightarrow B$ such that the following diagram

$$\begin{tikzcd} C \arrow[d, "\gamma"'] \arrow[dr, "u_i'"] & \\ B \arrow[r, "u_i"'] & A_i \end{tikzcd}$$

is commutative , for all i ([13], p.10).

In other words, intersection is the largest subobject of A contained in each A_i.

Remark 0.15. We denote the intersection by (B,u) and also as $B = \bigcap\limits_{i \in I} A_i$.

Dually, we define the cointersection of quotient objects $(C_i)_{i \in I}$ of an object B as the smallest quotient object containing in all C_i's.

Definition 0.20. A <u>category is said to have intersections/ cointersections</u> if every family of subobjects/ quotient objects has an intersection / cointersection.

0.1.6. <u>Kernels and cokernels</u>

Definition 0.21. Let $\mathcal{C}$ be a category with zero object. Then a morphism $u : K \longrightarrow A$ is <u>kernel</u> of a morphism $\alpha : A \longrightarrow B$ if

(i) $\alpha u = o$, and

(ii) if $\alpha u' = o$ for any other morphism $u' : K' \longrightarrow A$, then there exists a unique morphism $\Upsilon : K' \longrightarrow K$ such that

$$K' \xrightarrow{\Upsilon} K \xrightarrow{u} A = K' \xrightarrow{u'} A.$$

Definition 0.21*. Dually, we say that a morphism $p : B \longrightarrow C$ is <u>cokernel</u> of a morphism $\alpha : A \longrightarrow B$ if

(i) $p\alpha = o$, and

(ii) if $p'\alpha = o$ for any other morphism $B \longrightarrow C'$ there exists a unique morphism $\delta : C \longrightarrow C'$ such that

$$B \xrightarrow{p} C \xrightarrow{\delta} C' = B \xrightarrow{p'} C'.$$

Remark 0.16. Kernel α / cokernel α = equal (α, o)/coequal (α, o).

Definition 0.22. A category $\mathcal{C}$ <u>is said to have the kernels</u> / <u>cokernels</u> if every morphism in $\mathcal{C}$ has a kernel / cokernel in $\mathcal{C}$.

0.1.8. Natural categories

Definition 0.23. If a monomorphism $A' \xrightarrow{m} A$ is the kernel of some morphism in $\mathcal{C}$, then we say that A' is a normal subobject of A.

Definition 0.23*. Dually, if an epimorphism $A \xrightarrow{p} A''$ is cokernel of some morphism in $\mathcal{C}$, then A'' is called a conormal quotient object of A.

Definition 0.24. A category $\mathcal{C}$ is called normal category if every monomorphism in it is normal.

Definition 0.24*. Dually, if every epimorphism in a category is conormal, then the category is called conormal category ([13] , p.16).

Definition 0.25. Let $\mathcal{C}$ be a normal and conormal category with kernels and cokernels. We say that $\mathcal{C}$ is an exact category if every morphism $\alpha : A \longrightarrow B$ can be written as a composition $A \xrightarrow{q} I \xrightarrow{m} B$, where q is an epimorphi and m is a monomorphism.

Remark 0.17. Dual category of an exact category is exact.

A sequence of morphisms

$$\longrightarrow A_{i-1} \xrightarrow{\alpha_{i-1}} A_i \xrightarrow{\alpha_i} A_{i+1} \xrightarrow{\alpha_{i+1}} A_{i+2}$$

in an exact category is called an exact sequence if $\text{Ker}\ (\alpha_{i+1}) = I_m(\alpha_i)$ as a subobjects of A_{i+1} for every i .

Definition 0.26. An <u>additive category</u> is a category $\mathcal{C}$ together with an abelian structure on each morphism sets such that the following conditions are satisfied :

(i) The composition functions $[B,C] \times [A,B] \longrightarrow [A,C]$ are bilinear. That is , if α, $\beta \in [A,B]$ and $\gamma \in [B,C]$, then $\gamma(\alpha+\beta) = \gamma\alpha + \gamma\beta$ and if $\gamma \in [A,B]$ and α , $\beta \in [B,C]$ then $(\alpha+\beta)\gamma = \alpha\gamma + \beta\gamma$.

(ii) The zero element of abelian groups behaves as zero morphisms.

Definition 0.27. A category $\mathcal{C}$ is abelian if

(i) $\mathcal{C}$ has a zero object.

(ii) For every pair of objects, there is a product and a sum.

(iii) Every morphism has a kernel and cokernel.

(iv) Every monomorphism is a kernel of some morphism and every epimorphism is cokernel of some morphism.

Freyd proved :

Theorem 0.2. The following statements are equivalent

(a) $\mathcal{C}$ is an abelian category

(b) $\mathcal{C}$ has kernel, cokernel, finite products, finite coproducts and is normal and conormal.

(c) $\mathcal{C}$ has pushouts and pullbacks and is normal and conormal ([13] , p.35).

Now we shall define certain objects in a category :

0.1.9. <u>Projectivity and injectivity</u>

Definition 0.28. An object P in a category $\mathcal{C}$ is <u>projective</u> if and only if for every diagram

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] \\ A \arrow[r] & A'' \end{tikzcd}$$

with A ———> A'' an epimorphism there exists a unique morphism

$P \longrightarrow A$ making the diagram commutative.

Proposition 0.3. If $P = \bigoplus_{i \in I} P_i$ and if each P_i is projective, then P is projective . Conversely in a category with zero object , if P is projective then each P_i is projective ([2] , [3] , [13]).

Definition 0.29. An object Q in a category $\mathcal{C}$ is <u>injective</u> if and only if for every diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[dl, dashed] \\ Q & \end{tikzcd}$$

with $A' \longrightarrow A$ a monomorphism there exists a unique morphism $A \longrightarrow Q$ making the diagram commutative [13] .

Proposition 0.4. If $Q = \prod_{i \in I} Q_i$, then Q is injective iff each Q_i is injective.

Definition 0.30. An <u>essential extension</u> of an object A' is a monomorphism $A' \longrightarrow A$ such that for any nonzero subobject A_1 of A , $A' \cap A_1 \neq o$.

Theorem 0.3. A monomorphism $u : A' \longrightarrow A$ is an <u>essential</u> monomorphism if and only if every morphism $f : A \longrightarrow B$ such that fu is a monomorphism implies that f is a monomorphism [13] .

Remark 0.18. For our purpose , we shall consider theorem 0.3 as the definition of essential extension of A' or essential monomorphism .

Dually , we also have essential retraction.

Definition 0.31. An <u>injective envelope</u> for an object A is an essential extension $A \longrightarrow Q$ with Q injective.

Dually , we say that an essential retraction $P \longrightarrow A$ is <u>projective cover</u> of A if P is projective [13] .

Remark 0.19. The concept of essential extension and essential retraction are the generalisations of concepts of large and small respectively in the theory of modules.

0.1.9. <u>Generators and cogenerators</u>

Definition 0.32. A family of objects $\{U_i\}_{i \in I}$ is called a <u>family of generators</u> for a category $\mathcal{C}$ if for every pair of distinct morphisms $\alpha , \beta : A \longrightarrow B$, there exists a morphism

$u : U_i \longrightarrow A$ for some i such that $\alpha u \neq \beta u$.

Definition 0.32*. Dually, we define <u>family of cogenerators</u> if for every pair of distinct morphisms α, $\beta : A \longrightarrow B$ there exists a morphism $u : B \longrightarrow U_i$ for some i such that $u\alpha \neq u\beta$.

Remark 0.20. If the family $\{U_i\}_{i \in I}$ contains only one object U(say), then U is called generator / cogenerator in the category.

Theorem 0.4. Let $U = \bigoplus_{i \in I} U_i$ be in a category with zero object. Then U is generator if and only if each U_i is generator [1] .

0.1.10. <u>Reflectivity and coreflectivity</u>

Definition 0.33. Let $\mathcal{C}'$ be a subcategory of a category $\mathcal{C}$ and A be an object of $\mathcal{C}$. A <u>reflection</u> of A in $\mathcal{C}'$ is an object $R(A) \in \mathcal{C}'$ together with a morphism

$$\rho_A : R(A) \longrightarrow A$$

such that for every object $A' \in \mathcal{C}'$ and every morphism $A' \longrightarrow A$ there exists a unique morphism $A' \longrightarrow R(A)$ in $\mathcal{C}'$ such that the following diagram

$$\begin{tikzcd} A' \arrow[d] \arrow[dr, dashed] & \\ R(A) \arrow[r, "\rho_A"'] & A \end{tikzcd}$$

is commutative.

Definition 0.33*. Dually , $A \longrightarrow R(A)$ is called coreflection of A in $\mathcal{C}'$ if $R(A) \in \mathcal{C}'$ and if for any morphism $A \longrightarrow A'$ with $A' \in \mathcal{C}'$ there exists a unique morphism $R(A) \longrightarrow A'$ such that the following diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr, dashed] & R(A) \arrow[d] \\ & A' \end{tikzcd}$$

is commutative ([13] , p.128).

Definition 0.34. If each object A in $\mathcal{C}$ has a reflection / coreflection in the subcategory $\mathcal{C}'$, then $\mathcal{C}'$ is called reflective / coreflective subcategory of $\mathcal{C}$ ([13] , p.129)

0.2. Notions of functors

In this section , firstly , we give the concept of functors , covariant and contravariant , with their examples and later on we define certain special types of functors such as subfunctor , inclusion functor , identity functor , full , faithful , embedding , Hom , Tensor and adjoint functors etc. We also give certain results regarding the functions , which are useful for our further work.

0.2.1. Definitions and examples

Definition 0.35. Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. A covariant functor $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is an assignment of an object $T(A) \in \mathcal{C}'$ to each object $A \in \mathcal{C}$ and a morphism $T(\alpha) : T(A) \longrightarrow T(B)$ to each morphism $\alpha : A \longrightarrow B$ in $\mathcal{C}$ such that the following conditions are satisfied :

(i) Preservation of identities : For each object $A \in \mathcal{C}$, $T(I_A) = I_{T(A)}$.

(ii) Preservation of composition : If $\beta\alpha$ is defined in $\mathcal{C}$, then $T(\beta\alpha) = T(\beta)\, T(\alpha)$ ([13] , p.49).

The category $\mathcal{C}$ is called the domain of the functor T , and the category $\mathcal{C}'$ is called the codomain for T.

Definition 0.36. A <u>contravariant functor</u> $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is also an assignment of an object $T(A) \in \mathcal{C}'$ to each object $A \in \mathcal{C}$ and a morphism $T(\alpha) : T(B) \longrightarrow T(A)$ to each morphism $\alpha : A \longrightarrow B$ in $\mathcal{C}$ such that

(i) <u>Preservation of identities</u> : For each object $A \in \mathcal{C}$, $T(I_A) = I_{T(A)}$.

(ii) <u>Preservation of anticomposition</u> : If $\beta\alpha$ is defined in $\mathcal{C}$, then $T(\beta\alpha) = T(\alpha)\, T(\beta)$.

Remark 0.21. To each contravariant functor $T : \mathcal{C} \longrightarrow \mathcal{C}'$, a covariant functor from $\mathcal{C}^*$ into $\mathcal{C}'$ and a covariant functor from $\mathcal{C}$ into $\mathcal{C}'^*$ are associated in a natural manner, and vice versa ([1] , p.7). So that the general study of contravariant functors can always be reduced to the study of covariant functors.

Definition 0.37. A functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is a <u>subfunctor</u> of the functor $G : \mathcal{C} \longrightarrow \mathcal{C}'$ if $F(A) \leq G(A)$ for each object A of $\mathcal{C}$, and $F(f) = G(f)|F(A)$, precisely the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\quad F(f) \quad} & F(B) \\ \Big\downarrow{\scriptstyle i(A)} & & \Big\downarrow{\scriptstyle i(B)} \\ G(A) & \xrightarrow{\quad G(f) \quad} & G(B) \end{array}$$

is commutative , for all $f \in [A,B]_{\mathcal{C}}$, where $i(A)$ and $i(B)$ are inclusions.

Definition 0.38. The covariant functor $I : \mathcal{C} \longrightarrow \mathcal{C}$, such that $I_{\mathcal{C}}(A) = A$ for all $A \in \mathcal{C}$ and $I_{\mathcal{C}}(\alpha) = \alpha$ for all morphisms α in $\mathcal{C}$, is called the <u>identity functor</u> on $\mathcal{C}$.

Definition 0.39. Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$. Then a covariant functor $\mathcal{I} : \mathcal{C}' \longrightarrow \mathcal{C}$ such that $\mathcal{I}(A)=A$, $A \in \mathcal{C}'$ and $\mathcal{I}(\alpha) = \alpha$, $\forall \alpha \in \mathcal{C}'$, is called the <u>inclusion functor</u>.

Definition 0.40. A functor $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is said to be a <u>forgetful</u> functor if $T(A)$ forgets the structure associated with A , for all $A \in \mathcal{C}$.

<u>Example 0.13</u>. $F : \mathcal{A}b \longrightarrow$ Ens is a forgetful functor, which forgets the structure of being abelian , from the wategory of abelian groups to the category of sets.

<u>Other examples of functors</u>.

<u>Example 0.14</u>. Consider the mapping $F :$ Ens $\longrightarrow$ Gr , which associated to each set S the free group $F(S)$ generated by S. Since each function $f : S \longrightarrow S'$ in Ens can be extended to a unique group homomorphism $F(f): F(S) \longrightarrow F(S')$, yields a functor $F :$ Ens $\longrightarrow$ Gr , on the category of sets into the

category of groups.

<u>Example 0.15</u>. Let $\mathcal{C}$ be a category and A be a fixed object of $\mathcal{C}$. Then we have a covariant functor $H^A : \mathcal{C} \longrightarrow Ens$, with respect to A. Explicitly, for any $B \in \mathcal{C}$, $H^A(B) = [A, B]$ and if $\alpha : B \longrightarrow C$ then

$$H^A(\alpha) : [A , B] \longrightarrow [A, C]$$

is given by the rule $H^A(\alpha)(x) = \alpha x$, where $x \in [A, B]$.

If $\mathcal{C}$ is an additive category, then $H^A(B)$ is an abelian group and $H^A(\alpha)$ is a group homorphism. Hence H^A is a covariant functor from $\mathcal{C}$ into Gr.

Likewise, we have the contravariant functor $H_A : \mathcal{C} \longrightarrow Ens$ defined by $H_A(B) = [B, A]$ and $H_A(\alpha)x = x\alpha$ for $\alpha : B \longrightarrow C$ and $x \in [C, A]$. ([13] , p.50)

Remark 0.22. $(H^A)^* = H_{A^*}$. This gives the relation between H^A and H_A where A^* is the dual of A.

<u>Example 0.16</u>. Consider the category $\mathcal{M}_R$ of R-modules, where R is a commutative ring with identity and let M be a fixed R-module.

Define a function

$$T : \mathcal{M}_R \longrightarrow \mathcal{M}_R$$

as follows. For each module X over R in $\mathcal{M}_R$, let T(X) denote the tensor product $X \otimes M$ over R of the modules X and M. On the other hand , for each homomorphism $\alpha : X \longrightarrow Y$ of modules over R in $\mathcal{M}_R$, let $T(\alpha)$ denote the tensor product

$$T(\alpha) = \alpha \otimes i : X \otimes M \longrightarrow Y \otimes M .$$

for α and the identity endomorphism i of M. Then T is a covariant functor.

0.2.1. <u>Special types of functors</u>

Definition 0.41. Let $T : \quad \longrightarrow \quad '$ be a covariant functor from an additive category to an additive category ' . Then T is called <u>additive functor</u> if $T(\alpha + \beta) = T(\alpha) + T(\beta)$.

Definition 0.42. A covariant functor $T : \quad \longrightarrow \quad '$ is called a <u>mono/ epi functor</u> if $T(\alpha)$ is mono/epi in ' whenever α is mono/epi in .

Definition 0.43. A functor $T : \quad \longrightarrow \quad '$ is called <u>kernel preserving functor</u> if $T(u)$ is kernel of $T(\alpha)$ in ' when

$u : K \longrightarrow A$ is kernel of $\alpha : A \longrightarrow B$ in $\mathcal{C}$.

Definition 0.44. Let $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a covariant functor from an exact category $\mathcal{C}$ to an exact category $\mathcal{C}'$. Then we say that T is an <u>exact functor</u> if $T(A) \xrightarrow{T(\alpha)} T(B) \xrightarrow{T(\beta)} T(C)$ is exact in $\mathcal{C}'$ for the exact sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ in $\mathcal{C}$.

Definition 0.45. A covariant functor $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is called a <u>faithful functor</u> if for every pair of object A and B in $\mathcal{C}$ the function

$$[A, B] \longrightarrow [T(A), T(B)] \qquad \ldots(0.a)$$

induced by T is univariant (one to one).

Remark 0.23. If a functor $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is faithful , then it is called an <u>embedding</u> of $\mathcal{C}$ into $\mathcal{C}'$ in the sense of Freyd ([3] , p.66).

But in the sense of Mitchell [13] which is generally adopted we have the following :

Definition 0.46. A faithful functor which takes distinct objects into distinct objects is called on <u>embedding</u>.

Definition 0.47. A functor $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is said to be a <u>fullfunctor</u> if the map $[A,B] \longrightarrow [T(A),T(B)]$ induced by T is onto.

Definition 0.48. A functor $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is <u>representative</u> if for every object $A' \in \mathcal{C}'$, there is an object $A \in \mathcal{C}$ such that $T(A)$ and A' are isomorphic. A full representative , faithful functor is called an <u>equivalence</u> and the categories are said to be <u>equivalent categories</u>.

Remark 0.24. Upto now , we have dealt with the functors of one variable. The concept of functors of one variable can be generalized for the functors of more than one variable. The following are some of the examples :

<u>Example 0.17</u>. Let $\mathcal{M}_R$ be category of modules over a ring R. Then we define a functor Hom : $\mathcal{M}_R \times \mathcal{M}_R \longrightarrow \mathcal{M}_R$ as $\forall\ (A,B) \in \mathcal{M}_R \times \mathcal{M}_R$, Hom (A,B) is the set of all homomorphisms from A to B forming an abelian group in general. And , if $\alpha : A \longrightarrow B$, $\beta : C \longrightarrow D$, then

$$\text{Hom}\ (\alpha,\beta) : \text{Hom}\ (B,C) \longrightarrow \text{Hom}\ (A,D)$$

defined as Hom $(\alpha,\beta)(g) = \beta g\alpha$, $\forall\ g \in$ Hom (B,C).

Then Hom is a functor of two variables contravariant in the first and covariant in the second.

Similarly , we have the following example :

<u>Example 0.18</u>. Let $\mathcal{M}$ be the category whose objects are ordered triples $(R, A_R, {}_RB)$ where R is a ring, A_R is a right R-module and ${}_RB$ is a left R-module. A morphism of $(R, A_R, {}_RB)$ into $(R', A'_{R'}, {}_{R'}B')$ is , by definition , an ordered triple (ϕ, u, v), where (ϕ, u) is a bihomomorphism of the pair (R, A_R) into the pair $(R', A'_{R'})$ and (ϕ, v) is a bihomomorphism of $(R, {}_RB)$ into $(R', {}_{R'}B')$. Then we have a covariant functor T from the category $\mathcal{M}$ into the category $\mathcal{A}b$ of abelian groups if

$$T(R, A_R, {}_RB) = A \underset{R}{\otimes} B$$

and

$$T(\phi, u, v) = \text{the homomorphism induced by } (\phi, u, v)$$

from $A \underset{R}{\otimes} B \longrightarrow A' \underset{R'}{\otimes} B'$ ([1] , p.8).

In perticular , if we take R as a commutative ring , then we get a functor $\otimes : \mathcal{M}_R \otimes \mathcal{M}_R \longrightarrow \mathcal{A}b$, defined as

$$\otimes (A, B) = A \underset{R}{\otimes} B \quad \text{and}$$

$$\otimes\ (\alpha,\beta) : A \otimes_R B \longrightarrow A' \otimes_R B', \text{ for every } \alpha,\beta \in \mathcal{M}_R$$

such that

$$\otimes\ (\alpha,\beta)\ \ (a,b) = (\alpha \otimes \beta)(a,b) = \alpha(a) \otimes \beta(b).$$

This functor is covariant in both the variables.

0.2.3. <u>Natural transformation</u>

Definition 0.49. Let T_1, T_2 : $\mathcal{C} \longrightarrow \mathcal{C}'$ be two covariant functors and suppose that for every object $A \in \mathcal{C}$, we have a morphism $\eta_A : T_1(A) \longrightarrow T_2(A)$ in $\mathcal{C}'$ such that for every morphism $\alpha : A \longrightarrow B$ in $\mathcal{C}$ the diagram

$$\begin{tikzcd}
T_1(A) \arrow[r, "\eta_A"] \arrow[d, "T_1(\alpha)"'] & T_2(A) \arrow[d, "T_2(\alpha)"] \\
T_1(B) \arrow[r, "\eta_B"] & T_2(B)
\end{tikzcd}$$

is commutative. Then we say that η is a <u>natural transformation from T_1 to T_2</u> and denoted as $\eta : T_1 \longrightarrow T_2$.

If η_A is an isomorphism for each object A of $\mathcal{C}$ then η is called a natural equivalence, and we denote it as $\eta : T_1 \approx T_2$.

Proposition 0.5. A functor $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is an equivalence if and only if there is a functor $S : \mathcal{C}' \longrightarrow \mathcal{C}$ together with natural equivalence

$$\eta : I_{\mathcal{C}'} \longrightarrow TS \,,\; \phi : ST \to I_{\mathcal{C}} \;. \quad ([13]\,,p.61)$$

0.2.4. Remarks for construction of 'Cat' and 'Fonct'

Remark 0.25. Now we can have two special types of categories:

(i) Cat : Category of all categories, whose objects are categories and class of morphisms is the class of natural transformations between them.

(ii) Fonct : Category of functors : Let $\mathcal{C}$ be a small category and $\mathcal{C}'$ be another category. Then we have a category whose objects are all functors from $\mathcal{C}$ to $\mathcal{C}'$, and class of morphisms is the class of all natural transformations between them.

0.2.5. Adjoint functors

Definition 0.50. A covariant functor $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is said to be an adjoint for the covariant functor $S : \mathcal{C}' \longrightarrow \mathcal{C}$ if there is a natural equivalence of set-valued bifunctor (not necessarily unique)

$$\eta_{A',A} : [S(A'),A] \approx [A', T(A)] \qquad \ldots(0,b)$$

We also say that S is a coadjoint for T.

The adjoint situation given by natural equivalence defined in (0,b) above is also denoted as (η ,S,T , $\mathcal{C}$, $\mathcal{C}'$).

Given a natural equivalence as in (0,b) , for $A' \in \mathcal{C}'$, we denote by $\phi_{A'}$ the morphism

$$\eta_{A'S(A')}(I_{S(A')}) : A' \longrightarrow TSA' \qquad \ldots(0,c)$$

Dually, for $A \in \mathcal{C}$ we denote by ψ_A the morphism

$$\eta^{-1}_{T(A),A}(I_{T(A)}) : STA \longrightarrow A \qquad \ldots(0,d)$$

These two morphisms are required to define some special types of functors in Chapter II ahead.

References

Bucur [1], Eilenberg and Maclane [2] , Freyd [3], Gray [4] , Maclane [12], Mitchell [13], Zaidi [16] .

CHAPTER I

STRUCTURES ON $\mathcal{C}_S$ AND $\mathcal{C}^S$ IN RELATION TO $\mathcal{C}$.

1.0. Introduction. Maclane in 1965 [12] and Bucur-Deleanu [1] defined categories $\mathcal{C}_S$ and $\mathcal{C}^S$ as follows :

Definition 1.1. Let $\mathcal{C}$ be a category and S be an object of $\mathcal{C}$. Then the objects of $\mathcal{C}_S$ are pairs (A,α) , where A is an object of $\mathcal{C}$ and $\alpha : A \longrightarrow S$ a morphism in $\mathcal{C}$. The morphisms in $\mathcal{C}_S$ from an object (A,α) to an object (B,β) are taken to be the morphisms $\gamma : A \longrightarrow B$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} A \arrow[d, "\gamma"'] \arrow[dr, "\alpha"] & \\ B \arrow[r, "\beta"'] & S \end{tikzcd}$$

is commutative.

The composition of the morphisms is defined in the usual way.

Dually, the category $\mathcal{C}^S$ is defined as follows :

Definition 1.2. The objects of the category $\mathcal{C}^S$ are pairs of the form (α, A), where A is an object of $\mathcal{C}$ and $\alpha : S \longrightarrow A$ is a morphism in $\mathcal{C}$. The morphisms in $\mathcal{C}^S$ from an object (α, A) to an object (β, B) are the morphisms $\gamma : A \longrightarrow B$ in the category $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} & A \arrow[dd, "\gamma"] \\ S \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & \\ & B \end{tikzcd}$$

is commutative.

In this chapter , we study the categorical notions like products , coproducts, equalizers, coequalizers, intersections, cointersections, pullbacks, pushouts etc. in $\mathcal{C}_S$ and $\mathcal{C}^S$ and categorical structures like completeness , cocompleteness , filteredness, cofilteredness, normality, conormality and abelianness for $\mathcal{C}_S$ and $\mathcal{C}^S$ in relation to the original category $\mathcal{C}$. That is , if $\mathcal{C}$ has any categorical structure then we investigate whether it is preserved by $\mathcal{C}_S/\ \mathcal{C}^S$ or not. Generally , all the properties are preserved except that of product/coproduct which is not preserved. We find that it is also preserved if S is terminal/initial object. Also , if any property is contained by $\mathcal{C}_S$ or $\mathcal{C}^S$ then it is contained by $\mathcal{C}$ provided either S is terminal or initial according to the situation. In the last section of the chapter, we define V-category , in which certain concepts as pullbacks, intersections and some other structures like normality are preserved by $\mathcal{C}$, whenever they are preserved by $\mathcal{C}_S$ or $\mathcal{C}^S$ without any condition.

Generally , we shall prove the results for the case of $\mathcal{C}_S$ and results for $\mathcal{C}^S$ can be proved dually.

1.1. Initial and terminal objects in $\mathcal{C}_S$ and $\mathcal{C}^S$.

In this section , we study that if $\mathcal{C}$ has initial/terminal object then $\mathcal{C}_S$/ $\mathcal{C}^S$ also has initial/terminal object. But if $\mathcal{C}$ has terminal/initial object , then $\mathcal{C}_S$/ $\mathcal{C}^S$ may not have the terminal/initial object and hence if $\mathcal{C}$ has a zero object it is not necessary that $\mathcal{C}_S$ or $\mathcal{C}^S$ must have a zero object.

Proposition 1.1. If $\mathcal{C}$ has initial/terminal object , then $\mathcal{C}_S$ / $\mathcal{C}^S$ also has initial / terminal object.

Proof. Let I be an initial object of $\mathcal{C}$. Then the object (I,i) in $\mathcal{C}_S$ is an initial object in $\mathcal{C}_S$, where $i : I \longrightarrow S$ is unique morphism , since I is initial in $\mathcal{C}$. This follows atonce as there exist only one morphism $\Upsilon : I \longrightarrow A$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} I \arrow[dr, "i"] \arrow[dd, "\Upsilon"'] & \\ & S \\ A \arrow[ur, "\alpha"'] & \end{tikzcd}$$

is commutative , for all object (A,α) in $\mathcal{C}_S$.

Remark 1.1. If $\mathcal{C}$ has a zero object then $\mathcal{C}_S$ and $\mathcal{C}^S$ may not have zero object. For example , if $\mathcal{C}$ is the category of all groups with group homomorphisms. Then the trivial group is zero object in $\mathcal{C}$ and hence $\mathcal{C}_S$ has initial object $(0,z)$, where 0 is trivial group and $z : 0 \longrightarrow S$ is zero morphism , but it is not a terminal object in $\mathcal{C}_S$, because if we take (G',g) another object in $\mathcal{C}_S$ with a nonzero homomorphism $g : G' \longrightarrow S$, then there does not exist any nontrivial morphism $G' \longrightarrow 0$ such that the diagram

$$\begin{tikzcd} G' \arrow[dr, "g"] \arrow[dd, "?"] & \\ & S \\ 0 \arrow[ur, "z"'] & \end{tikzcd}$$

is commutative. This implies that $[(G',g),(0,z)] = \phi$. Hence, it is not a terminal object and, therefore, not a zero object.

Similarly, $(z,0)$, where $z : S \longrightarrow 0$, is not is an initial object for $\mathcal{C}^S$.

1.2. Characterization of products and coproducts in $\mathcal{C}_S$/ $\mathcal{C}^S$.

In this section, we study products and coproducts in $\mathcal{C}_S$ and $\mathcal{C}^S$ related to $\mathcal{C}$. Precisely, we observed that if $\mathcal{C}$ has coproducts/ products, then $\mathcal{C}_S$ / $\mathcal{C}^S$ has coproducts/ products. The converse is true only if S is terminal/initial object in $\mathcal{C}$. Further, if S is a terminal / initial object of $\mathcal{C}$, then $\mathcal{C}$ has products/ coproducts if and only if $\mathcal{C}_S$/ $\mathcal{C}^S$ has products / coproducts.

Lemma 1.1. If a category $\mathcal{C}$ has coproducts/ products, then so does $\mathcal{C}_S$ / $\mathcal{C}^S$.

Proof. Let $\{(A_i,\alpha_i)\}_{i \in I}$ be a family of objects in $\mathcal{C}_S$ hence $A_i \in \mathcal{C}$. Since $\mathcal{C}$ has coproducts, let $\{Q,u_i\}_{i \in I}$ be a coproduct of the family $\{A_i\}_{i \in I}$ in $\mathcal{C}$. Then there exists a unique morphism $\gamma : Q \longrightarrow S$ such that the following diagram

$$\begin{tikzcd}
A_i \arrow[rr, "u_i"] \arrow[dr, "\alpha_i"'] & & Q \arrow[dl, "\gamma"] \\
& S &
\end{tikzcd}$$

is commutative , for all $i \in I$. Therefore , we have $u_i : (A_i, \alpha_i) \longrightarrow (Q, \gamma)$ in $\mathcal{C}_S$.

Now we shall show that $\{(Q, \gamma), u_i\}_{i \in I}$ is coproduct of the given family in $\mathcal{C}_S$. It has already seen that $u_i : (A_i, \alpha_i) \longrightarrow (Q, \gamma)$ belongs to $\mathcal{C}_S$, $\forall i \in I$.

If we consider another family $\{(Q', \gamma'), u_i' \mid u_i' : (A_i, \alpha_i) \longrightarrow (Q', \gamma') \text{ in } \mathcal{C}_S\}_{i \in I}$, then we have the following diagram

$$\begin{tikzcd}
& & & Q' \arrow[dll] \\
A_i \arrow[urrr, "u_i'"] \arrow[rr, "u_i"] \arrow[dr, "\alpha_i"'] & & Q \arrow[dl, "\gamma"] \arrow[ur, dashed, "\delta"'] & \\
& S & &
\end{tikzcd}$$

Now , as Q is the coproduct of the family $\{A_i\}_{i\in I}$ in $\mathcal{C}$ there must exist a unique morphism $\delta : Q \longrightarrow Q'$ such that , $A_i \longrightarrow Q \longrightarrow Q' = A_i \longrightarrow Q'$, $\forall i \in I$. Since

$$\gamma'\delta u_i = \gamma' u_i' = \alpha_i = \gamma u_i \;, \quad \forall i \in I \;,$$

implies $\gamma'\delta = \gamma$, by the uniqueness of morphism in the definition of coproduct. Therefore , we have the unique morphism

$$\delta : (Q,\gamma) \longrightarrow (Q',\gamma')$$

in $\mathcal{C}_S$ such that

$$(A_i,\alpha_i) \xrightarrow{u_i} (Q,\gamma) \xrightarrow{\delta} (Q',\gamma') = (A_i,\alpha_i) \xrightarrow{u_i'} (Q',\gamma'),$$

$\forall i$. Hence $\{(Q,\gamma),\, u_i\}_{i\in I}$ is coproduct of the given family.

The pther part can be proved similarly

Now we prove the converse of the above lemma as follows:

Lemma 1.2. If $\mathcal{C}_S$ / $\mathcal{C}^S$ is a category with coproducts/products and S is a terminal/initial object in $\mathcal{C}$, then so is $\mathcal{C}$.

Proof. Let $\{A_i\}_{i \in I}$ be a family of objects in $\mathcal{C}$. Since S is a terminal object, there exist unique morphisms $\alpha_i : A_i \longrightarrow S$, $\forall i \in I$. So we have a family $\{(A_i, \alpha_i)\}_{i \in I}$ of objects in $\mathcal{C}_S$. Let $\{(Q,q), u_i\}_{i \in I}$ be coproduct in $\mathcal{C}_S$ of the family $\{(A_i, \alpha_i)\}_{i \in I}$. Then we claim that $\{Q, u_i\}_{i \in I}$ is the coproduct of the family $\{A_i\}_{i \in I}$ in $\mathcal{C}$. For, if we have another family $\{Q', u_i' \mid u_i' : A_i \longrightarrow Q'\}_{i \in I}$ then, since S is a terminal object, there exists a morphism $q' : Q' \longrightarrow S$ such that $A_i \xrightarrow{u_i'} Q' \xrightarrow{q'} S = A_i \xrightarrow{u_i} S$, $\forall i \in I$, implies $u_i' \in \mathcal{C}_S$, $\forall i \in I$. Thus, we have a family $\{(Q', q'), u_i'\}_{i \in I}$ in $\mathcal{C}_S$. Therefore, there exists a unique morphism $\gamma : (Q,q) \longrightarrow (Q',q')$ such that

$$(A_i, \alpha_i) \xrightarrow{u_i} (Q,q) \xrightarrow{\gamma} (Q',q') = (A_i, \alpha_i) \xrightarrow{u_i'} (Q',q'),$$

$\forall i \in I$, and hence we have a unique $\gamma : Q \longrightarrow Q'$ such that

$$A_i \xrightarrow{u_i} Q \xrightarrow{\gamma} Q' = A_i \xrightarrow{u_i'} Q',$$

$\forall i \in I$. This proves that $\mathcal{C}$ has coproducts. ||

The above two lemmas 1.1 and 1.2 give the following proposition :

Proposition 1.2. If $\mathcal{C}$ has coproducts / products , then $\mathcal{C}_S$ / $\mathcal{C}^S$ also has coproducts / products , and the converse is true if S is a terminal / initial object in $\mathcal{C}$. ||

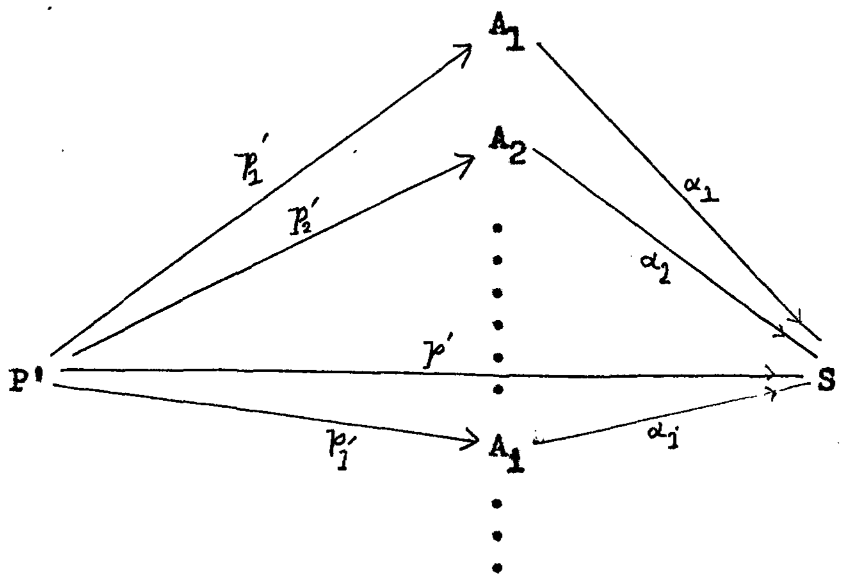
Now , we study the situation for products / coproducts in $\mathcal{C}_S$/ $\mathcal{C}^S$.

Lemma 1.3. If $\mathcal{C}$ has products / coproducts , then so does $\mathcal{C}_S$/ $\mathcal{C}^S$ provided that S is a terminal/initial object of $\mathcal{C}$.

Proof. Let $\{(A_i,\alpha_i)\}_{i\in I}$ be a family of objects in $\mathcal{C}_S$. Then we have a family $\{A_i\}_{i\in I}$ of objects in $\mathcal{C}$. Let $\{P,p_i\}_{i\in I}$ be the product of the family $\{A_i\}_{i\in I}$ in $\mathcal{C}$. Then we have the following diagram

$$\begin{tikzcd}
 & A_1 \arrow[ddr, "\alpha_1"] & \\
 & A_2 \arrow[dr, "\alpha_2"] & \\
P \arrow[uur, "p_1"] \arrow[ur, "p_2"] \arrow[dr, "p_i"'] & \vdots & S \\
 & A_i \arrow[ur, "\alpha_i"'] & \\
 & \vdots &
\end{tikzcd}$$

Since S is terminal object , $\alpha_1 p_1 = \alpha_2 p_2 = \cdots = \alpha_i p_i = \cdots \ \forall i$.
So , put $\alpha_i p_i = p$, $\forall i \in I$. Now we shall show that $\{(P,p), p_i\}_{i \in I}$ is product of given family in $\mathcal{C}_S$. Obviously $p_i \in \mathcal{C}_S, \forall i \in I$.
Let $\{(P',p'), p_i' \mid p_i' : (P',p') \longrightarrow (A_i,\alpha_i)\}_{i \in I}$ be another family of morphisms in $\mathcal{C}_S$. Then we have following diagram



By definition of product , there exists a unique map $\gamma : P' \longrightarrow P'$ in $\mathcal{C}$ such that $P' \xrightarrow{\gamma} P \xrightarrow{p_i} A_i = P' \xrightarrow{p_i'} A_i$, $\forall i \in I$.
Now , $p\gamma = \alpha_i p_i \gamma = \alpha_2 p_i' = p'$ implies that $\gamma : (P',p') \longrightarrow (P,p)$ is a unique morphism in $\mathcal{C}_S$ such that

$$(P',\gamma') \xrightarrow{\gamma} (P,\gamma) \xrightarrow{p_i} (A_i,\alpha_i) = (P',\gamma') \xrightarrow{p_i'} (A_i,\alpha_i)$$

$\forall i \in I$. This proves that $\mathcal{C}_S$ has products. ||

Lemma 1.4. Let S be a terminal/initial object in a category $\mathcal{C}$. Then , if $\mathcal{C}_S$ / $\mathcal{C}^S$ is category with products/coproducts then so is $\mathcal{C}$.

Proof. Let $\{A_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. Since S is a terminal object , there exist morphisms $\alpha_i : A_i \longrightarrow S$, $\forall i \in I$. Thus , we have a family $\{(A_i,\alpha_i)\}_{i\in I}$ of objects in $\mathcal{C}_S$. Let $\{(P,p),p_i\}_{i\in I}$ be product in $\mathcal{C}_S$ of the family $\{(A_i,\alpha_i)\}_{i\in I}$ in $\mathcal{C}_S$. Then we claim that $\{P,p_i\}_{i\in I}$ is the product of given family $\{A_i\}_{i\in I}$. For , if $\{P', p_i'\}_{i\in I}$ be another family of morphisms , and since S is a terminal object , there exists a morphism $p' : P' \longrightarrow S$ such that $P' \xrightarrow{p_i'} A_i \xrightarrow{\alpha_i} S = P' \xrightarrow{p'} S$, implies that $p_i \in \mathcal{C}_S$, $\forall i \in I$. So we have another family $\{(P',p'),p_i'\}_{i\in I}$ in $\mathcal{C}_S$. Hence , by definition of product , there exists a unique morphism $\gamma: (P',p') \longrightarrow (P,p)$ in $\mathcal{C}_S$ such that

$$(P',p') \xrightarrow{\gamma} (P,p) \xrightarrow{p_i} (A_i,\alpha_i) = (P',p') \xrightarrow{p_i'} (A_i,\alpha_i)$$

$\forall i \in I$. Hence , we have unique $\gamma : P' \longrightarrow P$ in $\mathcal{C}$ such that

$$P' \xrightarrow{\Upsilon} P \xrightarrow{p_i} A_i = P' \xrightarrow{p_i} A_i ,$$

$\forall i \in I$. This proves that $\mathcal{C}$ has products. ||

Lemmas 1.3 and 1.4 imply the following characterization :

Proposition 1.2. Let S be a terminal/initial object in $\mathcal{C}$. Then $\mathcal{C}$ has products / coproducts if and only if $\mathcal{C}_S/\mathcal{C}^S$ has products/coproducts.

The following theorem is a generalized statement of propositions 1.2 and 1.3.

Theorem 1.1. Let S be a terminal/initial object in the category $\mathcal{C}$. Then $\mathcal{C}$ has products and coproducts if and only if the category $\mathcal{C}_S$ / $\mathcal{C}^S$ has products and coproducts.

1.3. Characterization of equalizers and coequalizers in $\mathcal{C}_S/\mathcal{C}^S$.

In this section , we investigate that $\mathcal{C}_S$ and $\mathcal{C}^S$ have equalizers and coequalizers whenever $\mathcal{C}$ has equalizers and coequalizers respectively. But the converse is only true if S is terminal and initial object in for $\mathcal{C}_S$ and $\mathcal{C}^S$ respectively.

Lemma 1.5. If $\mathcal{C}$ has equalizers / coequalizers , then so does $\mathscr{C}_S \,/\, \mathscr{C}^S$.

Proof. Let γ_1 , γ_2 : $(A_1,\alpha_1) \longrightarrow (A_2,\alpha_2)$ be two morphisms in $\mathscr{C}_S$. Then γ_1, γ_2 : $A_1 \longrightarrow A_2$ are two morphisms in $\mathscr{C}$. Let $K \xrightarrow{\mu} A_1$ be equalizer of γ_1 and γ_2 in $\mathscr{C}$. Then we have a morphism $\gamma = \alpha_1\mu : K \longrightarrow S$ and hence $(K,\gamma) \in \mathscr{C}_S$ and $\mu : (K,\gamma) \longrightarrow (A_1,\alpha_1)$ in $\mathscr{C}_S$.

Now we shall show that $(K,\gamma) \xrightarrow{\mu} (A_1,\alpha_1)$ is the equalizers of γ_1 and γ_2 in $\mathscr{C}_S$.

Since $\gamma_1\mu = \gamma_2\mu$ in $\mathscr{C}$, hence in $\mathscr{C}_S$. Next , if $(K',\gamma') \xrightarrow{\mu'} (A_1,\alpha_1)$ be another morphism in $\mathscr{C}_S$ such that

$$(K',\gamma') \xrightarrow{\mu'} (A_1,\alpha_1) \xrightarrow{\gamma_1} (A_2,\alpha_2) = (K',\gamma') \xrightarrow{\mu'} (A_1,\alpha_1) \xrightarrow{\gamma_2} (A_2,\alpha_2)$$

That is , we have the following commutative diagram

$$\begin{tikzcd}
K' \arrow[rr, "\mu'"] \arrow[dr, dashed, "\delta"'] \arrow[drrr, "\gamma'"] & & A_1 \arrow[r, shift left, "\gamma_1"] \arrow[r, shift right, "\gamma_2"'] \arrow[dr, "\alpha_1"] & A_2 \arrow[d, "\alpha_2"] \\
& K \arrow[ur, "\mu"] \arrow[rr, "\gamma"'] & & S
\end{tikzcd}$$

That is , $\gamma_1\mu' = \gamma_2\mu'$ in $\mathcal{C}$. This implies that there exists a unique morphism $\delta : K' \longrightarrow K$ such that

$$K' \xrightarrow{\delta} K \xrightarrow{\mu} A_1 = K' \xrightarrow{\mu'} A_1.$$

Now, since $\gamma\delta = \alpha_1\mu\delta = \alpha_1\mu' = \gamma'$, therefore, $\gamma : (K',\gamma') \longrightarrow (K,\gamma) \in \mathcal{C}_S$ such that

$$(K',\gamma') \xrightarrow{\delta} (K,\gamma) \xrightarrow{\mu} (A_1,\alpha_1) = (K',\gamma') \xrightarrow{\mu'} (A_1,\alpha_1)$$

in $\mathcal{C}_S$. This proves that $\mathcal{C}_S$ has equalizers. ‖

On the other side , we have the following lemma :

Lemma 1.6. Let S be a terminal/initial object and $\mathcal{C}_S$/ $\mathcal{C}^S$ has equalizers/coequalizers. Then so does $\mathcal{C}$.

Proof. Let $\gamma_1, \gamma_2 : A_1 \longrightarrow A_2$ be two morphisms in $\mathcal{C}$. Since S is a terminal object , we have $\gamma_1,\gamma_2 : (A_1,\alpha_1) \longrightarrow (A_2,\alpha_2)$ in $\mathcal{C}_S$, where $\alpha_i : A_i \longrightarrow S$, $i = 1,2$; are unique morphisms. Let $\mu : (K,\gamma) \longrightarrow (A_1,\alpha_1)$ be equalizers of $\gamma_1,\gamma_2 : (A_1,\alpha_1) \longrightarrow (A_2,\alpha_2)$ in $\mathcal{C}_S$. Then we shall show that $\mu : K \longrightarrow A_1$ is equalizer of $\gamma_1,\gamma_2 : A_1 \longrightarrow A_2$ in $\mathcal{C}$.

Since $\gamma_1\mu = \gamma_2\mu$ in $\mathcal{C}_S$ hence in $\mathcal{C}$. Next if $K' \xrightarrow{\mu'} A_1$ be another morphism in $\mathcal{C}$ such that $\gamma_1\mu' = \gamma_2\mu'$ in $\mathcal{C}$. Now since S is a terminal object, there exists a unique morphism $k' : K' \longrightarrow S$ and hence we have a morphism

$$\mu' : (K',k') \longrightarrow (A_1,\alpha_1)$$

in $\mathcal{C}_S$ such that $\gamma_1\mu' = \gamma_2\mu'$ in $\mathcal{C}_S$. This implies that there exists a unique morphism $\delta : (K',k') \longrightarrow (K,k)$ in $\mathcal{C}_S$ such that

$$(K',k') \xrightarrow{\delta} (K,k) \xrightarrow{\mu} (A_1,\alpha_1) = (K',k') \xrightarrow{\mu'} (A_1,\alpha_1)$$

in $\mathcal{C}_S$. Therefore , we have unique morphism $\delta : K' \longrightarrow K$ in $\mathcal{C}$ such that

$$K' \xrightarrow{\delta} K \xrightarrow{\mu} A_1 = K' \xrightarrow{\mu'} A_1.$$

This proves that $\mathcal{C}$ has equalizers. ||

Lemmas 1.5 and 1.6 imply the following proposition :

Proposition 1.4. If $\mathcal{C}$ has equalizers/coequalizers then so does $\mathcal{C}_S / \mathcal{C}^S$ and the converse is true provided S is terminal/initial object in $\mathcal{C}$.

Now , we study the behaviour of coequalizers/equalizers in $\mathcal{C}_S$/ $\mathcal{C}^S$ in relation to $\mathcal{C}$.

Lemma 1.7. If $\mathcal{C}$ has coequalizers/equalizers , then so does $\mathcal{C}_S$ / $\mathcal{C}^S$.

Proof. Let $\gamma_1,\gamma_2 : (A_1,\alpha_1) \longrightarrow (A_2,\alpha_2)$ be two morphisms in $\mathcal{C}_S$. Then $\gamma_1,\gamma_2 : A_1 \longrightarrow A_2$ are in $\mathcal{C}$. Let $A_2 \xrightarrow{\mu} K$ be coequalizers of γ_1,γ_2 in $\mathcal{C}$. Now as $\gamma_1,\gamma_2 : (A_1,\alpha_1) \longrightarrow (A_2,\alpha_2)$ $\Longrightarrow$ $\alpha_2\gamma_1 = \alpha_2\gamma_2 = \alpha_1$. Therefore, by definition of coequalizers , there must exist a unique morphism $k: K \longrightarrow S$ such that

$$A_2 \xrightarrow{\mu} K \xrightarrow{k} S = A_2 \xrightarrow{\alpha_2} S \Longrightarrow \mu : (A_2,\alpha_2) \longrightarrow (K,k)$$

belongs to $\mathcal{C}_S$.

Now we shall show that $(A_2,\alpha_2) \xrightarrow{\mu} (K,k)$ is coequalizers of γ_1 and γ_2 in $\mathcal{C}_S$.

Since $\mu\gamma_1 = \mu\gamma_2$ in $\mathcal{C}$ hence in $\mathcal{C}_S$.

Next , let $(A_2,\alpha_2) \xrightarrow{\mu'} (K',k')$ be another morphism in $\mathcal{C}_S$ such that

$$(A_1,\alpha_1) \xrightarrow{\gamma_1} (A_2,\alpha_2) \xrightarrow{\mu'} (K',k') = (A_1,\alpha_1) \xrightarrow{\gamma_2} (A_2,\alpha_2) \xrightarrow{\mu'} (K',k$$

in $\mathcal{C}_S$. That is , we have the following commutative diagram :

$$\begin{tikzcd}
A_1 \arrow[r, shift left, "\gamma_1"] \arrow[r, shift right, "\gamma_2"'] \arrow[dr, "\alpha_1"'] & A_2 \arrow[r, "\mu"] \arrow[d, "\alpha_2"] \arrow[dr, "\mu'" near start] & K \arrow[dl, "k"' near start] \arrow[d, dotted, "\delta"] \\
 & S & K' \arrow[l, "k'"]
\end{tikzcd}$$

Since $\mu'\gamma_1 = \mu'\gamma_2$ in $\mathcal{C}_S$ hence in $\mathcal{C}$ ==> there exists a unique morphism $\delta : K \longrightarrow K'$ such that the whole above diagram is commutative.

Now, since $k'\delta\mu = k'\mu' = \alpha_2 = k\mu$ and $\mu : A_2 \longrightarrow K$ is coequalizer in $\mathcal{C}$, $k'\delta = k$. This implies that $\delta : (K,k) \longrightarrow (K',k')$ belongs to $\mathcal{C}_S$ and satisfies

$$(A_2,\alpha_2) \longrightarrow (K,k) \longrightarrow (K',k') = (A_2,\alpha_2) \longrightarrow (K',k').$$

This proves the lemma. ||

On the other side , we have :

Lemma 1.8. Let S be a terminal/initial object. Then, if $\mathcal{C}_S$ / $\mathcal{C}^S$ has coequalizers/equalizers then so does $\mathcal{C}$.

Proof. Let $\gamma_1, \gamma_2 : A_1 \longrightarrow A_2$ be two morphisms in $\mathcal{C}$. Since S is a terminal object, we have $\gamma_1, \gamma_2 : (A_1, \alpha_1) \longrightarrow (A_2, \alpha_2)$ in $\mathcal{C}_S$, $\alpha_i : A_i \longrightarrow S$, $i=1,2$. Let $(A_2, \alpha_2) \xrightarrow{\mu} (K, \gamma)$ be coequalizer of γ_1 and γ_2 in $\mathcal{C}_S$. Then we assert that $A_2 \xrightarrow{\mu} K$ is coequalizer of γ_1 and γ_2 in $\mathcal{C}$. Since $\mu\gamma_1 = \mu\gamma_2$ in $\mathcal{C}_S$ it holds in $\mathcal{C}$. Next, if $\mu' : A_2 \longrightarrow K'$ be another morphism in $\mathcal{C}$ such that $\mu'\gamma_1 = \mu'\gamma_2$ and S is a terminal object, there exists a unique morphism $\gamma' : K' \longrightarrow S$ such that we have the following diagram

$$\begin{tikzcd}
A_1 \arrow[r, shift left, "\alpha_1"] \arrow[r, shift right, "\alpha_2"'] \arrow[dr, "\alpha_1"'] & A_2 \arrow[r, "\mu"] \arrow[d, "\alpha_2"'] \arrow[dr, "\mu'"] & K \arrow[dl, "\gamma"'] \arrow[d, dashed, "\delta"] \\
& S & K' \arrow[l, "\gamma'"]
\end{tikzcd}$$

Again, since S is terminal object, $\gamma'\mu' = \alpha_2 \Longrightarrow \mu' \in \mathcal{C}_S$. This implies that there exists a unique morphism $\delta : (K, \gamma) \longrightarrow (K', \gamma')$ in $\mathcal{C}_S$ such that

$$(A_2, \alpha_2) \xrightarrow{\mu} (K, \gamma) \xrightarrow{\delta} (K8, \gamma') = (A_2, \alpha_2) \xrightarrow{\mu'} (K', \gamma')$$

Therefore , there exists a $\delta : K \longrightarrow K'$ in $\mathcal{C}$ such that

$$A_2 \xrightarrow{\mu} K \xrightarrow{\delta} K' = A_2 \xrightarrow{\mu'} K'.$$

This proves the lemma. ||

These two lemmas give the following proposition :

Proposition 1.5. If $\mathcal{C}$ has coequalizers/equalizers , then so does $\mathcal{C}_S$ / $\mathcal{C}^S$ and the converse is true if S is terminal/initial object in $\mathcal{C}$.

Proposition 1.4 and 1.5 can be strated together as follows :

Theorem 1.2. If $\mathcal{C}$ has equalizers and coequalizers, so does $\mathcal{C}_S$ / $\mathcal{C}^S$. Conversely , if S is a terminal/initial object in $\mathcal{C}$ and $\mathcal{C}_S$ / $\mathcal{C}^S$ has equalizers and coequalizers then $\mathcal{C}$ has also equalizers and coequalizers.

1.4. <u>Characterization of pullbacks and pushouts in $\mathcal{C}_S$/ $\mathcal{C}^S$</u>.

This section serves the study of pullbacks and pushouts in $\mathcal{C}_S$/ $\mathcal{C}^S$ in relation to $\mathcal{C}$. That is , we observe that if $\mathcal{C}$ has pullbacks or pushouts then so does $\mathcal{C}_S$ and $\mathcal{C}^S$ both , and the other way is true provided S is a terminal and initial object of the category respectively.

Lemma 1.9. If the category $\mathcal{C}$ has pushouts/pullbacks , then so has the category $\mathcal{C}_S$ / $\mathcal{C}^S$.

Proof. Consider the diagram

$$\begin{tikzcd}
(A,\alpha) \arrow[r, "\Upsilon_1"] \arrow[d, "\Upsilon_2"'] & (A_1,\alpha_1) \\
(A_2,\alpha_2) &
\end{tikzcd} \qquad \ldots(1.a)$$

in $\mathcal{C}_S$, which is the same as the following commutative diagram in $\mathcal{C}$.

$$\begin{tikzcd}
A \arrow[r, "\Upsilon_1"] \arrow[d, "\Upsilon_2"'] \arrow[dr, phantom, "\alpha"] & A_1 \arrow[d, "\Upsilon_1"] \\
A_2 \arrow[r, "\alpha_2"] & S
\end{tikzcd} \qquad \ldots(1.b)$$

Therefore , we have the following diagram

$$\begin{tikzcd}
A \arrow[r, "\Upsilon_1"] \arrow[d, "\Upsilon_2"'] & A_1 \\
A_2 &
\end{tikzcd} \qquad \ldots(1,c)$$

in $\mathcal{C}$.

Since $\mathcal{C}$ has pushouts , let the following commutative square

$$\begin{tikzcd} A \arrow[r, "\gamma_1"] \arrow[d, "\gamma_2"] & A_1 \arrow[d, "\beta_1"] \\ A_2 \arrow[r, "\beta_2"] & P \end{tikzcd} \qquad \ldots(1.d)$$

be pushout diagram of (1.e) in $\mathcal{C}$.

Now as diagram (1.b) is commutative, i.e. $\alpha_1\gamma_1 = \alpha_2\gamma_2$, there exists a unique morphism $p : P \longrightarrow S$ such that

$$A_i \xrightarrow{\beta_i} P \xrightarrow{p} S = A_i \xrightarrow{\alpha_i} S ,$$

i.e. $\gamma\beta_i = \alpha_i$, $i = 1,2$; $\Longrightarrow \beta_i : (A_i,\alpha_i) \longrightarrow (P,p), i=1,2$; are in $\mathcal{C}_S$ and $(P,p) \in \mathcal{C}_S$.

Thus , we have the following commutative square

$$\begin{tikzcd} (A,\alpha) \arrow[r, "\gamma_1"] \arrow[d, "\gamma_2"'] & (A_1,\alpha_1) \arrow[d, "\beta_1"] \\ (A_2,\alpha_2) \arrow[r, dash, "\beta_2"'] & (P,\alpha) \end{tikzcd} \qquad \ldots(1.e)$$

in $\mathcal{C}_S$. Now , we shall show that the diagram (1.e) is the pushout diagram for the diagram (1.a).

For this , let

$$\begin{tikzcd}
(A,\alpha) \arrow[r, "\gamma_1"] \arrow[d, "\gamma_2"'] & (A_1,\alpha_1) \arrow[d, "\beta_1'"] \\
(A_2,\alpha_2) \arrow[r, "\beta_2'"'] & (P',p')
\end{tikzcd} \qquad \ldots(1.f)$$

be another commutative square in $\mathcal{C}_S$ i.e., $\beta_1'\gamma_1=\beta_2'\gamma_2$ in $\mathcal{C}_S$, hence in $\mathcal{C}$. This implies that there exists a unique morphism $\gamma : P \longrightarrow P'$ such that

$$A_i \xrightarrow{\beta_i} P \xrightarrow{\gamma} P' = A_i \xrightarrow{\beta_i'} P' , \quad i=1,2.$$

Now , as $p'\gamma\beta_i = p'\beta_i' = \alpha_i = p\beta_i$, $i = 1 , 2$, and (1.d) is pushout for (1.c), we have $p'\gamma = p \Longrightarrow \gamma :(P,p)\longrightarrow(P',p')$ belongs to $\mathcal{C}_S$ such that

$$(A_i,\alpha_i) \xrightarrow{\beta_i} (P,p) \xrightarrow{\gamma} (P',p') = (A_i,\alpha_i) \xrightarrow{\beta_i'} (P',p'), \; i=1,2.$$

Thus $\mathcal{C}$ has pushouts. ||

Lemma 1.10. If S is a terminal/initial object and $\mathcal{C}_S$ / $\mathcal{C}^S$ has pushouts / pullbacks , then so does $\mathcal{C}$.

Proof. Let

$$\begin{array}{ccc} A & \xrightarrow{\gamma_1} & A_1 \\ {\scriptstyle\gamma_2}\downarrow & & \\ A_2 & & \end{array} \qquad \ldots(1.g)$$

be a diagram of morphisms in $\mathcal{C}$. Since S is a terminal object , we have the following diagram

$$\begin{array}{ccc} (A,\alpha) & \xrightarrow{\gamma_1} & (A_1,\alpha_1) \\ \downarrow{\scriptstyle\gamma_2} & & \\ (A_2,\alpha_2) & & \end{array} \qquad \ldots(1.h)$$

in $\mathcal{C}_S$, where $\alpha : A \longrightarrow S$, $\alpha_1 : A_1 \longrightarrow S$ are unique morphisms in $\mathcal{C}$. Now , since $\mathcal{C}_S$ has pushouts , let

$$\begin{array}{ccc} (A,\alpha) & \xrightarrow{\gamma_1} & (A_1,\alpha_1) \\ {\scriptstyle\gamma_2}\downarrow & & \downarrow{\scriptstyle\beta_1} \\ (A_2,\alpha_2) & \xrightarrow{\beta_2} & (P,p) \end{array} \qquad \ldots(1.i)$$

be pushout of the diagram (1.h) in $\mathcal{C}_S$. Then we have the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "\gamma_1"] \arrow[d, "\gamma_2"'] & A_1 \arrow[d, "\beta_1"] \\ A_2 \arrow[r, "\beta_2"] & P \end{tikzcd} \qquad \ldots(1.j)$$

in $\mathcal{C}$. Now we prove (1.j) is pushout for (1.g). For this, let

$$\begin{tikzcd} A \arrow[r, "\gamma_1"] \arrow[d, "\gamma_2"'] & A_1 \arrow[d, "\beta_1'"] \\ A_2 \arrow[r, "\beta_2'"] & P' \end{tikzcd}$$

be another commutative diagram in $\mathcal{C}$. Then , since S is terminal , we have the following commutative sequence

$$\begin{tikzcd} (A,\alpha) \arrow[r, "\gamma_1"] \arrow[d, "\gamma_2"'] & (A_1,\alpha_1) \arrow[d, "\beta_1'"] \\ (A_2,\alpha_2) \arrow[r, "\beta_2'"] & (P',p') \end{tikzcd}$$

in $\mathcal{C}_S$. Since (1.i) is pushout of (1.h) in $\mathcal{C}_S$, there exists a unique morphism $\gamma : (P,p) \longrightarrow (P',p')$ in $\mathcal{C}_S$ such that

$$(A_i,\alpha_i)\xrightarrow{\beta_i}(P,p)\xrightarrow{\gamma}(P',p')=(A_i,\alpha_i)\xrightarrow{\beta_i'}(P',p'),\ i=1,2.$$

Therefore, there is $\gamma : P \longrightarrow P'$ in $\mathcal{C}$ such that

$$A_i\xrightarrow{\beta_i}P\xrightarrow{\gamma}P' = A_i\xrightarrow{\beta_i'} P', \quad i=1,2.$$

This completes the proof. ||

The above two lemmas imply :

Proposition 1.6. If $\mathcal{C}$ has pushouts/pullbacks , then so does $\mathcal{C}_S$ / $\mathcal{C}^S$ and the converse is true provided S is terminal/initial object in the category $\mathcal{C}$.

Lemma 1.11. If $\mathcal{C}$ has pullbacks/ pushouts , then so does $\mathcal{C}_S$ / $\mathcal{C}^S$.

Proof. Consider the diagram

$$\begin{tikzcd} & (A_1,\alpha_1) \arrow[d, "\gamma_1"] \\ (A_2,\alpha_2) \arrow[r, "\gamma_2"] & (A,\alpha) \end{tikzcd} \qquad \ldots(1.k)$$

in $\mathcal{C}_S$. Then we have the diagram

$$\begin{tikzcd} & A_1 \arrow[d, "\gamma_1"] \\ A_2 \arrow[r, "\gamma_2"] & A \end{tikzcd} \qquad \ldots(1.\,l)$$

in $\mathcal{C}$. Since $\mathcal{C}$ has pullbacks , let

$$\begin{tikzcd} P \arrow[r, "\beta_1"] \arrow[d, "\beta_2"] & A_1 \arrow[d, "\gamma_1"] \\ A_2 \arrow[r, "\gamma_2"] & A \end{tikzcd} \qquad \ldots.(1.m)$$

be pullback diagram of (1. l) in $\mathcal{C}$, implying

$$\alpha_2\beta_2 = \alpha\,\gamma_2\beta_2 = \alpha\,\gamma_1\beta_1 = \alpha_1\beta_1$$

Therefore , we may consider $p = \alpha_1\beta_1 = \alpha_2\beta_2 : P \longrightarrow S$, which implies that $(P,p) \in \mathcal{C}_S$ and $\beta_i:(P,p) \longrightarrow (A_i,\alpha_i) \in \mathcal{C}_S$, $i = 1,2$; such that the following diagram

$$\begin{tikzcd} (P,p) \arrow[r, "\beta_1"] \arrow[d, "\beta_2"] & (A_1,\alpha_1) \arrow[d, "\gamma_1"] \\ (A_2,\alpha_2) \arrow[r, "\gamma_2"] & (A,\alpha) \end{tikzcd} \qquad \ldots(1.n)$$

is commutative in $\mathcal{C}_S$. Now, if we consider

$$\begin{array}{ccc} (P',p') & \xrightarrow{\beta_1} & (A_1,\alpha_1) \\ \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \gamma_1} \\ (A_2,\alpha_2) & \xrightarrow{\gamma_2} & (A,\alpha) \end{array} \qquad \ldots(1.o)$$

another commutative square in $\mathcal{C}_S$, it leads to a commutative square

$$\begin{array}{ccc} P' & \xrightarrow{\beta_1'} & A_1 \\ \downarrow{\scriptstyle \beta_2'} & & \downarrow{\scriptstyle \gamma_1} \\ A_2 & \xrightarrow{\gamma_2} & A \end{array} \qquad \ldots(1.p)$$

Since (1.m) is pullback square of (1. 1) , therefore , there exists a unique morphism $\gamma : P' \longrightarrow P$ in $\mathcal{C}$ such that

$$P' \xrightarrow{\gamma} P \xrightarrow{\beta_i} A_i = P' \xrightarrow{\beta_i'} A_i \ , \qquad i=1,2.$$

Now , $p\gamma = \alpha_i \beta_i \gamma = \alpha_i \beta_i' = p'$.

Then $\gamma : (P',p') \longrightarrow (P,p)$ is a unique morphism in $\mathcal{C}_S$.

such that

$$(P',p') \xrightarrow{\gamma} (P,p) \xrightarrow{\beta_1} (A_1,\alpha_1) = (P',p') \xrightarrow{\beta_1'} (A_1,\alpha_1)$$

This proves that (1.m) is the pullback diagram for (1.k). ||

Now, we shall check the reverse way of the lemma 1.11.

Lemma 1.12. Let S be a terminal/initial object and $\mathcal{C}_S/\mathcal{C}^S$ has pullbacks/ pushouts. Then so has $\mathcal{C}$.

Proof. Let

$$\begin{tikzcd} & A_1 \arrow[d, "\gamma_1"] \\ A_2 \arrow[r, "\gamma_2"] & A \end{tikzcd} \qquad \ldots(1.q)$$

be a diagram in $\mathcal{C}$. Since S is a terminal object, we have the following commutative diagram

$$\begin{tikzcd} & A_1 \arrow[d, "\gamma_1"'] \arrow[dr, "\alpha_1"] & \\ A_2 \arrow[r, "\gamma_2"] \arrow[rr, bend right, "\alpha_2"'] & A \arrow[r, "\alpha"] & S \end{tikzcd} \qquad \ldots(1.r)$$

in $\mathcal{C}$ and hence the following diagram

$$\begin{tikzcd} & (A_1,\alpha_1) \arrow[d, "\gamma_1"] \\ (A_2,\alpha_2) \arrow[r, "\gamma_2"] & (A,\alpha) \end{tikzcd} \qquad \ldots(1.s)$$

in $\mathcal{C}_S$. Since $\mathcal{C}_S$ has pullbacks, let the following commutative diagram

$$\begin{tikzcd} (P,p) \arrow[r, "\beta_1"] \arrow[d, "\beta_2"] & (A_1,\alpha_1) \arrow[d, "\gamma_1"] \\ (A_2,\alpha_2) \arrow[r, "\gamma_2"] & (A,\alpha) \end{tikzcd} \qquad \ldots(1.t)$$

be a pullback diagram of (1.s) in $\mathcal{C}_S$. Hence we have a commutative diagram

$$\begin{tikzcd} P \arrow[r, "\beta_1"] \arrow[d, "\beta_2"] & A_1 \arrow[d, "\gamma_1"] \\ A_2 \arrow[r, "\gamma_2"] & A \end{tikzcd} \qquad \ldots(1.u)$$

in $\mathcal{C}$. Now we shall show (1.u) is the pullback diagram of (1. 9). For this , let

$$\begin{tikzcd} P' \arrow[r, "\beta_1'"] \arrow[d, "\beta_2'"'] & A_1 \arrow[d, "\gamma_1"] \\ A_2 \arrow[r, "\gamma_2"] & A \end{tikzcd} \qquad \ldots(1.v)$$

be another commutative square in $\mathcal{C}$. Now as S is terminal object , we have

$$\begin{tikzcd} P' \arrow[r, "\beta_1'"] \arrow[d, "\beta_2'"'] \arrow[drr, "p'"] & A_1 \arrow[d, "\gamma_1"] \arrow[ddr, "\alpha_1"] & \\ A_2 \arrow[r, "\gamma_2"] \arrow[drr, "\alpha_2"'] & A \arrow[dr, "\alpha"] & \\ & & S \end{tikzcd} \equiv \begin{tikzcd} (P',p') \arrow[r, "\beta_1'"] \arrow[d, "\beta_2'"'] & (A_1,\alpha_1) \arrow[d, "\gamma_1"] \\ (A_2,\alpha_2) \arrow[r, "\gamma_2"] & (A,\alpha) \end{tikzcd}$$

a commutative square in $\mathcal{C}_S$. Now as, (1.t) is pullback square for (1.s) therefore there exists a unique morphism $\gamma : (P',p') \longrightarrow (P,p)$ such that

$$(P',p') \xrightarrow{\gamma} (P,p) \xrightarrow{\beta_i} (A_i,\alpha_i) = (P',p') \xrightarrow{\beta_i'} (A_i,\alpha_i), \quad i=1,2$$

in $\mathcal{C}_S$. Thus , we have a unique morphism $\gamma : P' \longrightarrow P$ in such that

$$P' \xrightarrow{\gamma} P \xrightarrow{\beta_i} A_i = P' \xrightarrow{\beta_i'} A_i, \; i=1,2. \qquad ||$$

The above two lemmas imply :

Proposition 1.7. If $\mathcal{C}$ has pullbacks/pushouts, then so does $\mathcal{C}_S / \mathcal{C}^S$, and the converse is true if S is terminal/initial object.

Proposition 1.6 and 1.7 lead to the following theorem :

Theorem 1.3. If $\mathcal{C}$ is a category with pullbacks and pushouts, then so is the category $\mathcal{C}_S / \mathcal{C}^S$, and the converse holds in case S is terminal/initial object in $\mathcal{C}$.

The following corollary is an immediate consequence of theorem 0.1 and theorem 1.3.

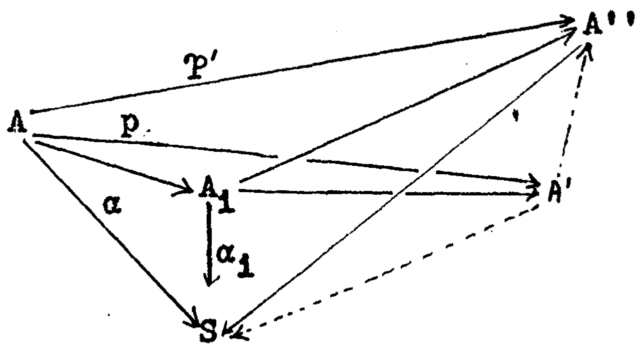
Corollary 1.1. If $\mathcal{C}$ has generalized pullbacks and generalized pushouts then $\mathcal{C}_S$ and $\mathcal{C}^S$ also have generalized pullbacks and pushouts. Conversely , if S is a terminal/initial object and $\mathcal{C}_S / \mathcal{C}^S$ has generalized pullbacks and generalized pushouts , then so has $\mathcal{C}$.

1.5. <u>Characterization of intersections and cointersections in $\mathcal{C}_S / \mathcal{C}^S$.</u>

This section is devoted to the study of intersections and cointersections in $\mathcal{C}_S / \mathcal{C}^S$ in relation to $\mathcal{C}$. We observe that if $\mathcal{C}$ has cointersections / intersections, then so has $\mathcal{C}_S / \mathcal{C}^S$ and the converse is true only if S is a terminal / initial object of $\mathcal{C}$. We also study the situation in $\mathcal{C}_S / \mathcal{C}^S$ for intersections/cointersections in a similar manner.

Lemma 1.13. If $\mathcal{C}$ has cointersections/intersections then so does $\mathcal{C}_S / \mathcal{C}^S$.

Proof. Let $\{(A,\alpha) \xrightarrow{p_i} (A_i,\alpha_i)\}_{i \in I}$ be a family of quotient objects of (A,α). Then $\{A \xrightarrow{p_i} A_i\}_{i \in I}$ is a family of quotient objects of A in $\mathcal{C}$. Let $A \xrightarrow{p} A'$ be cointersection of the family $\{A \xrightarrow{p_i} A_i\}_{i \in I}$ in $\mathcal{C}$. Then we have the following commutative diagram :



$\Longrightarrow \alpha_i p_i = \alpha$, $\forall\, i \in I$, by definition of morphisms in $\mathcal{C}_S$.

Since $A \xrightarrow{p} A'$ is the cointersection of p_i's in $\mathcal{C}$, there exists a unique morphism $\delta : A' \longrightarrow S$ in $\mathcal{C}$ such that $\delta p = \alpha = \alpha_i p_i$ or $\delta v_i p_i = \alpha_i p_i$, $\forall\, i \in I$, since p_i's are epi, $\delta v_i = \alpha_i$, $\forall\, i \in I$, $\Longrightarrow v_i : (A_i, \alpha_i) \longrightarrow (A', \delta)$ belongs to $\mathcal{C}_S$ for all $i \in I$.

Now , we shall show that $(A,\alpha) \xrightarrow{p} (A',\delta)$ is cointersection of the given family. For this , let $(A,\alpha) \xrightarrow{p'} (A'',\delta')$ be a morphism in $\mathcal{C}_S$ such that

$$(A,\alpha) \xrightarrow{p'} (A'',\delta') = (A,\alpha) \xrightarrow{p_i} (A_i,\alpha_i) \xrightarrow{v_i'} (A'',\delta')$$

in $\mathcal{C}_S$ and hence $p_i v_i' = p'$ in $\mathcal{C}$. Since $p : A \longrightarrow A'$ is a cointersection of the family $\left\{ A \xrightarrow{p_i} A_i \right\}_{i \in I}$, there exists a unique morphism $\mu : A' \longrightarrow A''$ such that $\mu v_i = v_i'$, $\forall\, i \in I$.

Now , $\delta' \mu v_i = \delta' v_i' = \alpha_i = \delta v_i$, $\forall\, i$, and since v_i's are epimirphisms , $\delta' \mu = \delta \Longrightarrow \mu : (A,\delta) \longrightarrow (A'',\delta')$ belongs to $\mathcal{C}_S$ such that

$$(A_i,\alpha_i) \xrightarrow{v_i} (A',\delta) \xrightarrow{\mu} (A'',\delta') = (A_i,\alpha_i) \xrightarrow{v_i'} (A'',\delta').$$

This proves the lemma. ||

The converse situation is given by the following lemma :

Lemma 1.14. If S is a terminal/initial object in $\mathcal{C}$ and $\mathcal{C}_S$ / $\mathcal{C}^S$ has cointersection/intersection , then so does $\mathcal{C}$.

Proof. Let $\{A \longrightarrow A_i\}_{i\in I}$ be a family of quotient objects of A in $\mathcal{C}$. Since S is terminal object , we have a family $\{(A,\alpha) \longrightarrow (A_i,\alpha_i) \mid \alpha : A \longrightarrow S \text{ and } \alpha_i : A_i \longrightarrow S \text{ in } \mathcal{C}\}_i$ of quotients in $\mathcal{C}_S$. Let $(A,\alpha) \xrightarrow{p} (A',\alpha')$ be cointersection of this family in $\mathcal{C}_S$. Then

$$(A,\alpha) \xrightarrow{p} (A',\alpha') = (A,\alpha) \xrightarrow{p_i} (A_i,\alpha_i) \xrightarrow{v_i} (A',\alpha')$$

in $\mathcal{C}_S$, and hence $A \xrightarrow{p} A' = A \xrightarrow{p_i} A_i \xrightarrow{v_i} A'$, $\forall i\in I$, in $\mathcal{C}$. Now , we show that $A \xrightarrow{p} A'$ is cointersection of the given family in $\mathcal{C}$. For this , let $A \xrightarrow{p} A''$ be another morphism in $\mathcal{C}$ such that $A \xrightarrow{p'} A'' = A \xrightarrow{p_i} A_i \xrightarrow{v_i'} A''$. Then, since S is terminal , there exists a morphism $\alpha'' : A'' \longrightarrow S$,

$$(A,\alpha) \xrightarrow{p'} (A'',\alpha'') = (A,\alpha) \xrightarrow{p_i} (A_i,\alpha_i) \xrightarrow{v_i'} (A'',\alpha'')$$

in $\mathcal{C}_S$. Therefore , there exists a unique morphism $\delta : (A',\alpha') \longrightarrow (A'',\alpha'')$ such that

$$(A_i,\alpha_i) \xrightarrow{v_i'} (A'',\alpha'') = (A_i,\alpha_i) \xrightarrow{v_i} (A',\alpha') \xrightarrow{\delta} (A'',\alpha'') .$$

Hence , there is a morphism $\delta: A' \longrightarrow A''$ in $\mathcal{C}$ such that

$$A_i \xrightarrow{v_i'} A'' = A_i \xrightarrow{v_i} A' \xrightarrow{\delta} A''$$

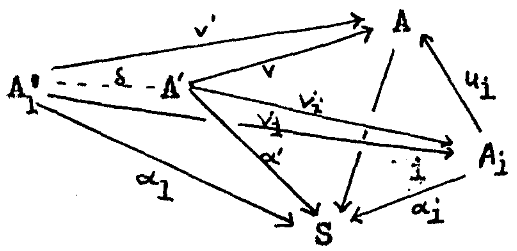
This proves the lemma. ||

The above two lemmas 1.13 and 1.14 imply the following proposition :

Proposition 1.8. If a category $\mathcal{C}$ has cointersections/intersections then so has $\mathcal{C}_S$ / $\mathcal{C}^S$, and the converse is true provided S is a terminal/initial object in $\mathcal{C}$.

Lemma 1.15. If $\mathcal{C}$ has intersections/cointersections , then $\mathcal{C}_S$/ $\mathcal{C}^S$ also has intersectionsp/ cointersections.

Proof. Let $\{(A_i,\alpha_i) \xrightarrow{u_i} (A,\alpha)\}_{i\in I}$ be a family of subobjects of (A,α) in $\mathcal{C}_S$. Then $\{A_i \xrightarrow{u_i} A\}_{i\in I}$ is a family of subobjects of A in $\mathcal{C}$. Let $A' \xrightarrow{u} A$ be intersection of the family $\{A_i \xrightarrow{u_i} A\}_{i\in I}$ in $\mathcal{C}$. Then we have the following commutative diagram



This implies $\alpha_i v_i = \alpha u_i v_i = \alpha u$, $i \in I$. So , considering $\alpha' = \alpha_i v_i : A' \longrightarrow S$, $(A',\alpha') \in \mathcal{C}_S$ and $u:(A',\alpha')\longrightarrow(A,\alpha)\in\mathcal{C}_S$

Now , we shall show that $u : (A',\alpha') \longrightarrow (A,\alpha)$ is the intersection of the given family. For this, let $(A_1',\alpha_1') \xrightarrow{u'} (A,\alpha)$ be another morphism in $\mathcal{C}_S$ such that

$$(A_1',\alpha_1') \xrightarrow{u'} (A,\alpha) = (A_1',\alpha_1') \xrightarrow{v_i'} (A_i,\alpha_i) \xrightarrow{u_i} (A,\alpha) ,$$

$i \in I$, in $\mathcal{C}_S$ and hence , we have

$$A_1' \xrightarrow{u'} A = A_1' \xrightarrow{v_i'} A_i \xrightarrow{u_i} A$$

in $\mathcal{C}$. Now , since $A' \xrightarrow{u} A$ is an intersection of the family $\{A_i \longrightarrow A\}_{i\in I}$, there exists a unique morphism $\delta : A_i' \longrightarrow A'$ such that

$$A_i' \xrightarrow{\delta} A' \xrightarrow{v_i} A_i = A_i' \xrightarrow{v_i'} A_i \ , \ \forall i \in I \ , \text{ in } \mathcal{C}.$$

Now $\alpha'\delta = \alpha_i v_i \delta = \alpha_i v_i' = \alpha_i'$ implies that there exists $\delta : (A_i' , \alpha_i') \longrightarrow (A', \alpha')$ in $\mathcal{C}_S$ such that

$$(A_i',\alpha_i') \xrightarrow{\delta} (A',\alpha') \xrightarrow{v_i} (A_i,\alpha_i) = (A_i',\alpha_i') \xrightarrow{v_i'} (A_i,\alpha_i)$$

in $\mathcal{C}_S$. Then completes the proof. ||

Conversely , we have the following :

Lemma 1.16. If S is a terminal/initial object in the category $\mathcal{C}$ and $\mathcal{C}_S$ /$\mathcal{C}^S$ has intersections/cointersections , then so has $\mathcal{C}$.

Proof. Let $\{A_i \xrightarrow{u_i} A\}_{i\in I}$ be a family of subobjects of an object A in the category $\mathcal{C}$. Since S is terminal, we have a family $\{(A_i,\alpha_i) \xrightarrow{u_i} (A,\alpha)\}_{i\in I}$ of subobjects of (A,α) in $\mathcal{C}_S$. Now , as $\mathcal{C}_S$ has intersections , let $(A',\alpha') \xrightarrow{u} (A,\alpha)$

be the intersection of the family in $\mathcal{C}_S$. Then we have $u : A' \longrightarrow A$ in $\mathcal{C}$ such that $A' \xrightarrow{u} A = A' \xrightarrow{v_i} A_i \xrightarrow{u_i} A$.

To prove that $A' \xrightarrow{u} A$ is an intersection of the given family in $\mathcal{C}$, consider a morphism $A'' \xrightarrow{u'} A$ in $\mathcal{C}$ such that

$$A'' \xrightarrow{u'} A = A'' \xrightarrow{v_i'} A_i \xrightarrow{u_i} A \ , \ \forall i \in I \ , \quad \text{in } \mathcal{C} \ .$$

Since S is a terminal object in , we have

$$(A'',\alpha'') \xrightarrow{u'} (A,\alpha) = (A'',\alpha'') \xrightarrow{v_i'} (A_i,\alpha_i) \xrightarrow{u_i} (A,\alpha),$$

$\forall i \in I$, in $\mathcal{C}_S$. Therefore , by definition of intersection, there exists a unique morphism $\delta : (A'',\alpha'') \longrightarrow (A',\alpha')$ in $\mathcal{C}_S$, such that

$$(A'',\alpha'') \xrightarrow{\delta} (A',\alpha') \xrightarrow{v_i} (A_i,\alpha_i) = (A'',\alpha'') \xrightarrow{v_i'} (A_i,\alpha_i)$$

Hence , we have the unique $\delta : A'' \longrightarrow A'$ in $\mathcal{C}$ such that

$$A'' \xrightarrow{v_i'} A_i = A'' \xrightarrow{\delta} A' \xrightarrow{v_i} A_i \ , \ \forall i \in I \ \text{ in } \mathcal{C}.$$

This completes the proof of the lemma. ||

The above two lemmas lead to the following proposition :

Proposition 1.9. If $\mathcal{C}$ has intersections/cointersections, then so has $\mathcal{C}_S/\mathcal{C}^S$, and the converse is true if S is terminal/initial object in $\mathcal{C}$. ||

The above two propositions 1.8 and 1.9 give the following theorem :

Theorem 1.4. If a category $\mathcal{C}$ has intersections and cointersections, then $\mathcal{C}_S$ and $\mathcal{C}^S$ have also intersections and cointersections. Conversely, if S is a terminal/initial object in the category $\mathcal{C}$ and the category $\mathcal{C}_S/\mathcal{C}^S$ has intersections and cointersections then so has $\mathcal{C}$. ||

1.6. Conditions for completeness and cocompleteness of $\mathcal{C}_S$ and $\mathcal{C}^S$.

In this section, we introduce the notions of completeness and cocompleteness in the categories above and below objects and determine the conditions under which these categories are complete or cocomplete in relation to $\mathcal{C}$. We first give the preliminary notions involved.

Definition 1.3. A diagram scheme Σ is a triplet (I,M,d), where I is a set whose elements are called <u>vertices</u> , M is a set whose elements are called <u>arrows</u> , and d is a function from M to $I \times I$. If $m \in M$ and $d(m) = (i,j)$, then we call i the <u>origin</u> of m and j the <u>extremity</u> of m. A <u>diagram</u> in a category $\mathcal{C}$ over the scheme Σ is a function D which assigns to each vertex $i \in I$ an object D_i of $\mathcal{C}$ and to each arrow m with origin i and extremity j , a morphism $D(m) \in [D_i , D_j]$ ([13] , p. 42).

Definition 1.4. If D is a diagram in a category $\mathcal{C}$ over the diagram scheme $\Sigma = (I,M,d)$, X,Y are any objects in $\mathcal{C}$, we call a family of morphisms $\{X \xrightarrow{\alpha_i} D_i\}_{i \in I}$ a <u>compatible</u> family for D if for every arrow $m \in M$ the diagram

$$\begin{tikzcd} & D_i \arrow[d, "D(m)"] \\ X \arrow[ur, "\alpha_i"] \arrow[r, "\alpha_j"'] & D_j \end{tikzcd}$$

is commutative. This family is called a <u>limit</u> for D if it is compatible , and if for every compatible family $\{Y \longrightarrow D_i\}_{i \in I}$, there is a unique morphism $Y \longrightarrow X$ such that , for each $i \in I$, we have $Y \longrightarrow X \longrightarrow D_i = Y \longrightarrow D_i$, or the following diagram is commutative

$$\begin{tikzcd}
 & & D_i \arrow[dd, "D(m)"] \\
Y \arrow[r, dashed] \arrow[urr, "\alpha'_i"] \arrow[drr, "\alpha'_j"'] & X \arrow[ur, "\alpha_i"'] \arrow[dr, "\alpha_j"] & \\
 & & D_j
\end{tikzcd}$$

Dually , we have the concept of colimit of D. Mitchell [13] has proved the following proposition .

Proposition 1.10. A category $\mathcal{C}$ is <u>complete</u> iff it has products and finite intersections , dually , it is <u>cocomplete</u> iff it has coproducts and finite cointersections.

The above proposition 1.10 , with proposition 1.2, and 1.8 leads us to the following theorem .

Theorem 1.5. A category $\mathcal{C}_S$ / $\mathcal{C}^S$ is cocomplete /complete if $\mathcal{C}$ is cocomplete/complete. The converse is true only if S is a terminal/initial object in $\mathcal{C}$.

We also have the following theorem.

Theorem 1.6. Let S be a terminal/initial object of $\mathcal{C}$. Then

$\mathcal{C}_S$ / $\mathcal{C}^S$ is a complete/cocomplete integory if and only if $\mathcal{C}$ is a complete/cocomplete category.

1.7. Conditions for $\mathcal{C}_S$ and $\mathcal{C}^S$ being filtered and cofiltered categories

A filtered category is defined by H.B.Stauffer [15] . In this section , we again observe the same type of phenomenon in $\mathcal{C}_S$ and $\mathcal{C}^S$ as we have been observing in other categorial structure. In the converse situation the existance of terminal/initial object in $\mathcal{C}$ is as important.

Definition 1.5. A category $\mathcal{C}$ is called filtered category if it satisfies :

(i) For any two objects A and B of $\mathcal{C}$, there exist an object D in $\mathcal{C}$ and morphisms $D \longrightarrow A$, $D \longrightarrow B$ in $\mathcal{C}$.

(ii) For each pair of morphisms $\alpha, \beta : A \longrightarrow B$ in $\mathcal{C}$, there exists a morphisms $D \xrightarrow{\gamma} A$ in $\mathcal{C}$ such that

$$D \xrightarrow{\gamma} A \xrightarrow{\alpha} B = D \xrightarrow{\gamma} A \xrightarrow{\beta} B$$

Dullay, cofiltered category is likewise defined.

The following lemma follows trivially by definitions of Products, coproducts, equalizers, coequalizers, and filtered and cofiltered category.

Lemma 1.24. A category $\mathcal{C}$ is filtered/cofiltered if $\mathcal{C}$ has products/coproducts and equalizers/coequalizers .

Now, we have the following theorem , which follows immediately from propositions 1.3, 1.9 and lemma 1.24.

Theorem 1.7. A category $\mathcal{C}_S$/ $\mathcal{C}^S$ is filtered/cofiltered if the category $\mathcal{C}$ is filtered/cofiltered. Conversely, if S is a terminal/ an initial object of the category $\mathcal{C}$, then $\mathcal{C}$ is filtered/cofiltered if $\mathcal{C}_S$/ $\mathcal{C}^S$ is a filtered/cofiltered category.

1.8. Conditions for $\mathcal{C}_S$ and $\mathcal{C}^S$ being normal and conormal categories

In this section , we find that if S is a terminal object, then $\mathcal{C}$ is normal if and only if $\mathcal{C}_S$ is normal similar type of results hold for conormality. For this , we first investigate the behaviour of mono and epimorphisms in relation to $\mathcal{C}$, and how they are preserved.

Lemma 118. If $f : (A_1,\alpha_1) \longrightarrow (A_2,\alpha_2) / (\alpha_1,A_1) \longrightarrow (\alpha_2,A_2)$ is a monomorphism/epimorphism in $\mathcal{C}_S / \mathcal{C}^S$, then $f: A_1 \longrightarrow A_2$ is a monomorphism /epimorphism in the category $\mathcal{C}$, further if $f \in \mathcal{C}_S$ and is a monomorphism/epimorphism in the category $\mathcal{C}$, then it is monomorphism/epimorphism in the category $\mathcal{C}_S / \mathcal{C}^S$.

Proof. Let $C \xrightarrow{\gamma_1} A_1 \xrightarrow{f} A_2 = C \xrightarrow{\gamma_2} A_1 \longrightarrow A_2$ in $\mathcal{C}$. Since $f : (A_1,\alpha_1) \longrightarrow (A_2,\alpha_2)$, we have the following commutative diagrams

$$\begin{tikzcd} C \arrow[r, "\gamma_1"] & A_1 \arrow[r, "f"] \arrow[d, "\alpha_1"'] & A_2 \arrow[dl, "\alpha_2"] \\ & S & \end{tikzcd} = \begin{tikzcd} C \arrow[r, "\gamma_2"] & A_1 \arrow[r, "f"] \arrow[d, "\alpha_1"'] & A_2 \arrow[dl, "\alpha_2"] \\ & S & \end{tikzcd}$$

$f\gamma_1 = f\gamma_2 \Longrightarrow \alpha_2 f\gamma_1 = \alpha_2 f\gamma_2 \Longrightarrow \alpha_1\gamma_1 = \alpha_1\gamma_2 = \alpha : C \longrightarrow S$ (say) $\Longrightarrow \gamma_1, \gamma_2 \in \mathcal{C}_S$ and we have

$$(C,\alpha) \xrightarrow{\gamma_1} (A_1,\alpha_1) \xrightarrow{f} (A_2,\alpha_2) = (C,\alpha) \xrightarrow{\gamma_2} (A_1,\alpha_1) \xrightarrow{f} (A_2,\alpha_2)$$

in $\mathcal{C}_S$. Since f is mono in $\mathcal{C}_S$, $\gamma_1 = \gamma_2$ in $\mathcal{C}_S$ hence in $\mathcal{C}$ $\Longrightarrow f$ is monomorphism in $\mathcal{C}$.

Other part is obvious because , if $\Upsilon_1 f = \Upsilon_2 f$ in $\mathcal{C}_S$ hence in $\mathcal{C}$ $\Longrightarrow \Upsilon_1 = \Upsilon_2$.

Lemma 1.19. If S is a terminal/an initial object of the category $\mathcal{C}$ and $f \in \mathcal{C}_S / \mathcal{C}^S$ is epi/monomorphism in $\mathcal{C}_S / \mathcal{C}^S$, then f is epi/monomorphism in the category $\mathcal{C}$. Fürther , an epi/monomorphism in $\mathcal{C}$, which is a morphism in $\mathcal{C}_S / \mathcal{C}^S$, is epi/monomorphism in $\mathcal{C}_S / \mathcal{C}^S$.

Proof. Let $A_1 \xrightarrow{f} A_2 \xrightarrow{\Upsilon_1} B = A_1 \xrightarrow{f} A_2 \xrightarrow{\Upsilon_2} B$ in $\mathcal{C}$. Since $f \in \mathcal{C}_S$, we have the following commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r, "f"] \arrow[dr, "\alpha_1"'] & A_2 \arrow[r, "\Upsilon_1"] \arrow[d, "\alpha_2"] & B & = & A_1 \arrow[r, "f"] \arrow[dr, "\alpha_1"'] & A_2 \arrow[r, "\Upsilon_2"] \arrow[d, "\alpha_2"] & B \\
& S & & & & S &
\end{tikzcd}$$

As S is terminal object , there exists a unique morphism $B \xrightarrow{\delta} S$ such that $\delta\Upsilon_1 = \delta\Upsilon_2 = \alpha_2 \Longrightarrow \Upsilon_1, \Upsilon_2 : (A_2, \alpha_2) \longrightarrow (B, \delta) \in$ Since f is epi in $\mathcal{C}_S$ and hence $\Upsilon_1 f = \Upsilon_2 f$ in $\mathcal{C}_S$ implies $\Upsilon_1 = \Upsilon_2$ in $\mathcal{C}_S$ and hence in $\mathcal{C}$. The other part is easily follows. ||

Remark 1.1. This lemma supplies the information that epimorphism in $\mathcal{C}$ is not induced from that in $\mathcal{C}_S$ unless S is a terminal object , whereas the converse happens without this restriction. At the same time , the situation is different for $\mathcal{C}^S$, whenever epi in $\mathcal{C}^S$ is epi in $\mathcal{C}$.

Proposition 1.11. Let S be a terminal / an initial object of a category $\mathcal{C}$. Then $\mathcal{C}$ is normal/conormal if and only if $\mathcal{C}_S$ / $\mathcal{C}^S$ is normal/conormal.

Proof. Let $\mathcal{C}$ be a normal category and let $f:(A_1,\alpha_1) \dashrightarrow (A_2,\alpha_2) \in \mathcal{C}_S$ be a monomorphism. Then, by lemma 1.18, $f : A_1 \dashrightarrow A_2$ is mono in $\mathcal{C}$, [illegible].18 hence it should be kernel of some morphism $\Upsilon : A_2 \dashrightarrow B$ (say) in $\mathcal{C}$ as $\mathcal{C}$ is normal. Now, as S is a terminal object , Υ will belong to $\mathcal{C}_S$, hence f is kernel of Υ in $\mathcal{C}_S \Longrightarrow \mathcal{C}_S$ is normal.

Conversely , let $\mathcal{C}_S$ be a normal category and $f:A_1 \dashrightarrow A_2$ be a monomorphism in $\mathcal{C}$. Then , as S is terminal object , $f \in \mathcal{C}_S$ and is monomorphism by lemma 1.18, hence by hypothesis it is kernel of some morphism Υ in $\mathcal{C}_S$, thus in $\mathcal{C} \Longrightarrow \mathcal{C}$ is normal.

Dually , we have the following :

Proposition 1.12. Let S be a terminal/ an initial object in $\mathcal{C}$. Then $\mathcal{C}$ is conormal/normal if and only if $\mathcal{C}_S$/ $\mathcal{C}^S$ is conormal/normal. ||

1.9. <u>$\mathcal{C}_S$ and $\mathcal{C}^S$ as abelian categories</u>

This section gives a characterization for abelianness of $\mathcal{C}_S$ and $\mathcal{C}^S$ in relation to $\mathcal{C}$. And this characterization directly follows from the propositions 1.2, 1.3, 1.11 and 1.12 and the following proposition proved by Freyd (Mitchell [13]) p.33)

Proposition 1.13. A category $\mathcal{C}$ is abelian if and only if $\mathcal{C}$ has pushouts , pullbacks and is normal and conormal. ||

We obtain the following theorem for abelianness of the categories $\mathcal{C}_S$ and $\mathcal{C}^S$ as a consequence of Proposition refered to.

Theorem 1.8. Let S be a terminal/an initial object in $\mathcal{C}$. Then $\mathcal{C}$ is abelian if and only if $\mathcal{C}_S$ / $\mathcal{C}^S$ is abelian.

1.10. <u>V-categories and $\mathcal{C}_S$ / $\mathcal{C}^S$</u>

First we define a V-category as follows :

Definition 1.6. A category $\mathcal{C}$ will be called a V-category if

$[A,B] \neq \emptyset$ for all objects A and B in $\mathcal{C}$.

Example 1.1. Most of the categories are V-categories,like the category of sets with functions, category of groups with group homomorphisms , category of rings with ring homomorphisms, category of modules with module homomorphisms etc., in genrral all categories of algebraic structures with corresponding morphisms.

The category P.O.set, whose objects are members of the partially ordered set and morphisms are defined as follows

$$[a,b] = \text{singleton} \quad \text{if } a \leq b$$

$$[a,b] = \emptyset \qquad \text{if } a > b ,$$

is not a V-category and also has no initial and terminal object , But there are categories , which are not V-categories but have initial and terminal objects, eg : if we replace P.O.set by bounded P.O.set in the above example, then it will be a category , which is not a V-category , but has terminal and initial objects,as upper and lower bounded respectively.

As we have observed so far that if $\mathcal{C}$ has some property

P (say) , then $\mathcal{C}_S$ and $\mathcal{C}^S$, in general, possess the same property P , but conversely if $\mathcal{C}_S$ and $\mathcal{C}^S$ hold some properties then those properties are possessed by $\mathcal{C}$ only if S is either terminal or initial object. Now we shall show that we can remove the restriction of S being terminal or initial object in some cases if the category , we consider , is a V-category for instance we can do it in the case of pullbacks/pushouts and intersections / cointersections. The corresponding lemmas , in this direction , are as follows :

Lemma 1.20. Let $\mathcal{C}$ be a V-category and $\mathcal{C}_S$ / $\mathcal{C}^S$ has pullbacks/ pushouts. Then $\mathcal{C}$ has pullbacks/pushouts.

Proof. Suppose $\mathcal{C}_S$ has pullbacks and let

$$\begin{tikzcd} & A_1 \arrow[d, "\gamma_1"] \\ A_2 \arrow[r, "\gamma_2"] & A \end{tikzcd}$$

be a diagram in $\mathcal{C}$. As $\mathcal{C}$ is a V-category , there always exists a morphism say $\alpha : A \longrightarrow S$ in $\mathcal{C}$, then considering $\alpha_1 = \alpha\, \gamma_1 : A_1 \longrightarrow S$ and $\alpha_2 = \alpha\, \gamma_2 : A_2 \longrightarrow S$, we have the following diagram in $\mathcal{C}_S$

$$\begin{array}{ccc} & & (A_1,\alpha_1) \\ & & \downarrow{\scriptstyle \gamma_1} \\ (A_2,\alpha_2) & \xrightarrow{\gamma_2} & (A,\alpha) \end{array}$$

Also , since $\mathcal{C}_S$ has pullbacks , let

$$\begin{array}{ccc} (P,p) & \xrightarrow{\beta_1} & (A_1,\alpha_1) \\ \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \gamma_1} \\ (A_2,\alpha_2) & \xrightarrow{\gamma_2} & (A,\alpha) \end{array}$$

be a pullback diagram in $\mathcal{C}_S$. Now, proceeding as in lemma 1.12 we see that the commutatice diagram :

$$\begin{array}{ccc} P & \xrightarrow{\beta_1} & A_1 \\ \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \gamma_1} \\ A_2 & \xrightarrow{\gamma_2} & A \end{array}$$

is the pullback of the considered diagram in $\mathcal{C}$.

Similarly, we can prove the following lemma :

Lemma 1.21. Let $\mathcal{C}$ be a V-category and $\mathcal{C}_S/\ \mathcal{C}^S$ has intersections/cointersections. Then $\mathcal{C}$ also has intersections/ cointersections.

Remark 1.2. If $\mathcal{C}$ is a V-category without a terminal or initial objects then even if $\mathcal{C}_S/\ \mathcal{C}^S$ has (i) products and coproducts (ii) coequalizers/equalizers (iii) pushouts/pullbacks (iv) cointersections/intersections , it is not necessary that $\mathcal{C}$ may have the same properties. The same will be true if $\mathcal{C}$ is not a V-category and S is not a terminal/an initial object. For (i) we have no suitable objects in $\mathcal{C}_S\ /\ \mathcal{C}^S$ and for (ii),(iii) and (iv) , we have no suitable morphisms.

We now show that if $\mathcal{C}$ is a V-category , then the converse of Proposition 1.11 holds , without restriction of terminal/initial objects.

Lemma 1.22. If a monomorphism/ an epimorphism f in $\mathcal{C}_S/\ \mathcal{C}^S$ is the kernel /cokernel of some morphism g in $\mathcal{C}_S/\ \mathcal{C}^S$, then f is the kernel/cokernel of g in $\mathcal{C}$.

Proof. Since $(A,\alpha)\xrightarrow{f}(B,\beta)\xrightarrow{g}(C,\gamma)=0$ in $\mathcal{C}_S$ and hence

$A \xrightarrow{f} B \xrightarrow{g} C = 0$ in $\mathcal{C}$, Also f is monomorphism in $\mathcal{C}$ by lemma 1.18. Now , let

$$A' \xrightarrow{f'} B \xrightarrow{g} C = 0 \text{ in } \mathcal{C} \Longrightarrow$$

we have

$$(A', \beta f') \xrightarrow{f'} (B, \beta) \xrightarrow{g} (C, \gamma) = 0 \text{ in } \mathcal{C}_S$$

i.e.

$$\begin{tikzcd}
A' \arrow[drr, "f'"] \arrow[dr, dashed, "\delta"] \arrow[ddrr, "\alpha'"'] & & & \\
& A \arrow[r, "f"] \arrow[dr, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[dl, "\gamma"] \\
& & S &
\end{tikzcd}$$

$\Longrightarrow$ there exists a unique morphism $\delta : (A', \beta f') \longrightarrow (A, \alpha)$ such that

$$(A', \beta f') \xrightarrow{\delta} (A, \alpha) \xrightarrow{f} (B, \beta) = (A', \beta f') \xrightarrow{f'} (B, \beta)$$

and hence there exists a unique $\delta : A' \longrightarrow A$ in such that $A' \longrightarrow A \longrightarrow B = A' \longrightarrow B \Longrightarrow f$ is kernel of g in $\mathcal{C}$.

Proposition 1.14. Let $\mathcal{C}$ be a V-category and $\mathcal{C}_S / \mathcal{C}^S$ is normal/conormal. Then $\mathcal{C}$ is normal/conormal.

Proof. Let $\mathcal{C}_S$ be a normal and $f : A \longrightarrow B$ be a monomorphism in $\mathcal{C}$. As $\mathcal{C}$ is a V-category , there exists $\beta : B \longrightarrow S$. Now consider $\alpha = \beta f : A \longrightarrow S$ then we have $(A,\alpha) \xrightarrow{f} (B,\beta)$ in $\mathcal{C}_S$ and is a monomorphism in $\mathcal{C}_S$ by lemma 1.17 , $\Longrightarrow$ there exists a $g : (B,\beta) \longrightarrow (C,\gamma)$ in $\mathcal{C}_S$ such that f is kernel of g in $\mathcal{C}_S$ and hence, by lemma 1.22 , f is kernel of g in $\mathcal{C}$ $\Longrightarrow$ $\mathcal{C}$ is normal. ||

References

Bucur [1], Freyd [3] , Maclane [12] , Mitchell [13], Stauffer [15] , Zaidi [16] .

CHAPTER II

FUNCTORS ON $\mathcal{C}_S$ AND $\mathcal{C}^S$

2.0. Introduction. In this chapter, we define certain special functors with the help of the categories $\mathcal{C}_S$ and $\mathcal{C}^S$ and study them in different ways. Firstly, we define forgetful functors F / F' on $\mathcal{C}_S / \mathcal{C}^S$ and obtain the equivalence of $\mathcal{C}$ with $\mathcal{C}_S$ and $\mathcal{C}^S$; and then, for a given morphism $f : S \longrightarrow S'$ in $\mathcal{C}$, obtain special types T_f and T^f, also T_{cat} and T^{cat} induced by T_f and T^f. Furthermore, we induce certain other functors from given functor $T : \mathcal{C} \longrightarrow \mathcal{C}'$ and investigate their relative properties.

2.1. Equivalence of categories $\mathcal{C}$, $\mathcal{C}_S$ and $\mathcal{C}^S$

In this section, we first define a forgetful functor F from $\mathcal{C}_S$ to $\mathcal{C}$, and further considering that S is a terminal object, we define another functor $G : \mathcal{C} \longrightarrow \mathcal{C}_S$ and obtain that $FG = I$ and $GF = I$, which implies that $\mathcal{C} \approx \mathcal{C}_S$; In a similar way, we get $\mathcal{C} \approx \mathcal{C}^S$ if S is an initial object.

Let $\mathcal{C}$ be a category and S be an object of $\mathcal{C}$ and if $\mathcal{C}_S$ and $\mathcal{C}^S$ are categories as defined in chapter I, then we have forgetful functor F/F'.

Definition 2.1.

$$F : \mathcal{C}_S \longrightarrow \mathcal{C}$$

such that

$$F(A,\alpha) = A, \quad (A,\alpha) \in \mathcal{C}_S$$

and

$$F(f) = f, \quad \text{morphism } f \in \mathcal{C}_S$$

Definition 2.2. Define

$$F' : \mathcal{C}^S \longrightarrow \mathcal{C}$$

such that

$$F'(\alpha,A) = A, \quad (\alpha,A) \in \mathcal{C}^S$$

and

$$F'(f) = f, \quad \text{morphism } f \in \mathcal{C}^S$$

Both F and F' are covariant forgetful functors,which forget the morphisms in the pairs denoting the objects in $\mathcal{C}_S$ / $\mathcal{C}^S$.

Again , let S be a terminal object in $\mathcal{C}$, define the functor G.

Definition 2.3.

$$G : \mathcal{C} \longrightarrow \mathcal{C}_S$$

such that

$$G(A) = (A,\alpha), \text{where } \alpha : A \longrightarrow S \text{ is a unique morphism in } \mathcal{C}.$$

and

$$G(f) = f \quad , \text{for all } f : A \longrightarrow B \text{ in } \mathcal{C} \text{ and also}$$
$$f : (A,\alpha) \longrightarrow (B,\beta) \text{ in } \mathcal{C}_S \text{ because,}$$
$$S \text{ is a terminal object.}$$

If S is an initial object , defined likewise G'.

Definition 2.4.

$$G' : \mathcal{C} \longrightarrow \mathcal{C}^S \quad .$$

such that

$$G'(A) = (\alpha,A), \text{where } \alpha : S \longrightarrow A \text{ is a unique morphism in } \mathcal{C}.$$

and

$$G'(f) = f \quad , \text{ for all } f : A \longrightarrow B \text{ in } \mathcal{C} \text{ and also}$$
$$f : (\alpha,A) \longrightarrow (\beta,B) \text{ in } \mathcal{C}^S \text{ because}$$
$$S \text{ is an initial object in } \mathcal{C}.$$

It can be easily seen that both G and G' are covariant functors.

Proposition 2.1. If S is a terminal/an initial object of a category $\mathcal{C}$, then $\mathcal{C}$ is equivalent to $\mathcal{C}_S$ / $\mathcal{C}^S$.

Proof. Let S be a terminal object and A be an object of $\mathcal{C}$.

Then

$FG(A) = F(A,\alpha)=A=I_{\mathcal{C}}(A)$, where, $I_{\mathcal{C}}$ denotes the identity functor from $\mathcal{C} \longrightarrow \mathcal{C}$

$GF(A,\alpha)=G(A)=(A,\alpha)=I_{\mathcal{C}_S}(A,\alpha)$, where $I_{\mathcal{C}_S}$ denotes the identity functor from $\mathcal{C}_S \longrightarrow \mathcal{C}_S$.

Similarly,

$$FG(f) = I_{\mathcal{C}}(f), \quad f \in \mathcal{C}$$

$$GF(f) = I_{\mathcal{C}_S}(f), \quad f \in \mathcal{C}_S$$

$\Longrightarrow$ $FG = I_{\mathcal{C}}$ and $GF = I_{\mathcal{C}_S}$

$\Longrightarrow$ $\mathcal{C} \approx \mathcal{C}_S$ //.

2.2. Functors T_f and T^f

In this section, we define functors $T_f : \mathcal{C}_S \longrightarrow \mathcal{C}_{S'}$ and $T^f : \mathcal{C}^{S'} \longrightarrow \mathcal{C}^S$, for a morphism $f : S \longrightarrow S'$ in a category $\mathcal{C}$ and observe that these functors are additive if $\mathcal{C}$ is additive.

Definition 2.5. Let $\mathcal{C}$ be a category and $f : S \longrightarrow S'$ be a morphism in $\mathcal{C}$. Then we have a twofold map :

$$T_f : \mathcal{C}_S \longrightarrow \mathcal{C}_{S'}$$

such that

(i) $T_f(A,\alpha) = (A, f\alpha)$, for all objects (A,α) in $\mathcal{C}_S$

(ii) $T_f(\lambda) = \lambda$, where $\lambda : (A,\alpha) \longrightarrow (B,\beta) \in \mathcal{C}_S$.

This definition is valid as λ also belongs to $\mathcal{C}_{S'}$,
since the following diagram is commutative

$$\begin{tikzcd}
A \arrow[rr, "\lambda"] \arrow[dr, "\alpha"] \arrow[ddr, "f\alpha"'] & & B \arrow[dl, "\beta"'] \arrow[ddl, "f\beta"] \\
& S \arrow[d, "f"] & \\
& S' &
\end{tikzcd}$$

Obviously , T_f is a covariant functor , because

$$T_f(I_{(A,\alpha)}) = I_{(A,\alpha)} = I_{T_f(A,\alpha)}$$

$$T_f(\lambda\mu) = \lambda\mu = T_f(\lambda)\ T_f(\mu) .$$

Definition 2.6. Also similarly we define another twofold map

$$T^f : \mathcal{C}^{S'} \longrightarrow \mathcal{C}^S$$

such that

$$T^f(\alpha,A)=(\alpha f,A), \text{for all objects } (\alpha,A) \text{ in } \mathcal{C}^{S'}$$

and if $\lambda : (\alpha,A) \longrightarrow (\beta,B)$ be a morphism in $\mathcal{C}^{S'}$, then

$$T^f(\lambda) = \lambda \in \mathcal{C}^S .$$

The definition is valid as the following diagram is commutative.

$$\begin{tikzcd}
& S \arrow[d, "f"] \arrow[ddl, "\alpha f"'] \arrow[ddr, "\beta f"] & \\
& S' \arrow[dl, "\alpha"] \arrow[dr, "\beta"'] & \\
A \arrow[rr, "\lambda"] & & B
\end{tikzcd}$$

T^f is again a covariant functor.

Proposition 2.2. If and $\mathcal{C}_S$/ $\mathcal{C}^{S'}$ are additive categories , then T_f/ $T^{f'}$ is an additive functor.

The proof is evident since ,

$$T_f(\lambda+\mu) = \lambda+\mu = T_f(\lambda) + T_f(\mu) .$$

Definition 2.7. Let $\mathcal{C}$ be an additive category and T_f and $T_{f'}$

be two functors from $\mathcal{C}_S$ to $\mathcal{C}_{S'}$. Then we define ,
(i) addition of T_f and $T_{f'}$, such that

$$(T_f+T_{f'})\ (A,\alpha)=T_f(A,\alpha)+T_{f'}(A,\alpha), \qquad \ldots(2.a)$$

$$\forall \text{ objects } (A,\alpha) \in \mathcal{C}_S$$

and

$$(T_f+T_{f'})\ (\lambda) = T_f(\lambda) + T_{f'}(\lambda), \qquad \ldots(2.b)$$

$$\forall\, \lambda \in \mathcal{C}_S .$$

(ii) addition of two objects in $\mathcal{C}_S$, such that

$$(A,\alpha) + (A,\beta) = (A\ ,\ \alpha + \beta\ ) \qquad \ldots(2.c)$$

2.3. Functors T_{cat} and $T^{cat} : \mathcal{C} \longrightarrow \mathcal{C}at$

Let $\mathcal{C}$ be a category and $\mathcal{C}at$ be the category of all categories. Then we defined functors $T_{cat} : \mathcal{C} \longrightarrow \mathcal{C}at$ and also $T^{cat} : \mathcal{C} \longrightarrow \mathcal{C}at$ and find out that T_{cat} is covariant and T^{cat} is contravariant (additive) functors. Also T_{cat} is left limit preserving whereas functors T^{cat} takes right limit to left limit.

Definition 2.8. Let $\mathcal{C}$ be a category and $\mathcal{C}at$ be the category of all categories. Define a twofold map

$$T_{cat} : \mathcal{C} \longrightarrow \mathcal{C}at$$

such that

(i) $T_{cat}(S) = \mathcal{C}_S$, for all objects S in $\mathcal{C}$, and

(ii) if $f : S \longrightarrow S'$ be a morphism in $\mathcal{C}$,

then

$T_{cat}(f) = T_f : \mathcal{C}_S \longrightarrow \mathcal{C}_{S'}$, where T_f is as defined in previous section 2.2.

Likewise, define twofold map T^{cat} as follows :

Definition 2.9.

$$T^{cat} : \mathcal{C} \longrightarrow \mathcal{C}at$$

such that

(i) $T^{cat}(S) = \mathcal{C}^S$, for all objects S in $\mathcal{C}$, and

(ii) if $f: S \longrightarrow S'$ be a morphism in $\mathcal{C}$, then

$$T^{cat}(f) = T^f : \mathcal{C}^{S'} \longrightarrow \mathcal{C}^S,$$

where T^f is as defined in section 2.2.

Lemma 2.1. The twofold map T_{cat} is a covariant functor.

Proof. From (i) of definition 2.8, T_{cat} takes an object of $\mathcal{C}$ to an object of $\mathcal{C}at$, and by (ii) of definition 2.8, T_{cat} takes a morphism of $\mathcal{C}$ to an morphism of $\mathcal{C}at$ (i.e. a functor from the category $\mathcal{C}_S$ to $\mathcal{C}_{S'}$). Also , for identity morphism $I_S : S \longrightarrow S$,

$$T_{cat}(I_S) = T_{I_S} = I_{\mathcal{C}_S} = I_{T_{cat}(S)}.$$

To check covariance , let $f : S \longrightarrow S'$ and $f' : S' \longrightarrow S''$ be two morphisms in $\mathcal{C}$ and (A,α) be an object in $\mathcal{C}_S$. Then

$$\begin{aligned}
[T_{cat}(f'f)]\ (A,\alpha) &= T_{f'f}\ (A,\alpha) = (A,f'f\alpha) \\
&= T_{f'}(A,f\alpha) \\
&= T_{f'}(T_f(A,\alpha)) \\
&= T_{f'}T_f(A,\alpha) \\
&= [T_{cat}(f')\ T_{cat}(f)]\ (A,\alpha)
\end{aligned}$$

$\Longrightarrow T_{cat}(f'f) = T_{cat}(f')\ T_{cat}(f).$

Hence , T_{cat} is a covariant functor. ||

Lemma 2.2. The twofold map $T^{cat} : \mathcal{C} \longrightarrow \mathcal{C}at$ is a contravariant functor.

Proof. By definition 2.9, T^{cat} takes an object of $\mathcal{C}$ to an object of $\mathcal{C}at$ and a morphism of $\mathcal{C}$ to a morphism of $\mathcal{C}at$ (i.e. a functor from $\mathcal{C}^{S'} \longrightarrow \mathcal{C}^{S}$). Also , for identity morphism $I_S : S \longrightarrow S$, we have

$$T^{cat}(I_S) = T^{I_S} = I_{\mathcal{C}^S} = I_{T^{cat}(S)} .$$

To check contravariance , let $S \xrightarrow{f} S', S' \xrightarrow{f'} S''$ be two morphisms in $\mathcal{C}$ and (A,α) be an object in $\mathcal{C}^{S''}$. Then

$$\begin{aligned}
[T^{cat}(f'f)]\ (\alpha,A) &= T^{f'f}(\alpha,A) \\
&= (\alpha\ f'f,A) \\
&= T^{f}(\alpha f',A) \\
&= T^{cat}(f)\ (\alpha f',A) \\
&= T^{cat}(f)\ (T^{f'}(\alpha,A)) \\
&= T^{cat}f\ (T^{cat}(f')\ (\alpha,A)) \\
&= [T^{cat}(f)\ T^{cat}(f')]\ (\alpha,A)
\end{aligned}$$

$\Longrightarrow T^{cat}(f'f)=T^{cat}(f)T^{cat}(f')$.

Hence T^{cat} is a contravariant functor. ||

Lemma 2.3. If $\mathcal{C}$ is an additive category , then T_{cat} and T^{cat} are additive functors.

Proof. Let f and f' be two morphisms from S to S' in $\mathcal{C}$ i.e. $f,f' : S \longrightarrow S'$ and (A,α) be an object of $\mathcal{C}_S$. Then

$$\begin{aligned}
T_{cat}(f+f')\ (A,\alpha) &= T_{f+f'}(A,\alpha) \\
&= (A,(f+f')\alpha) \\
&= (A,f\alpha+f'\alpha) && \text{since } \mathcal{C} \text{ is additive} \\
&= (A,\alpha f)+(A,f'\alpha) && \text{by definition 2.7 of additivity of objects in } \mathcal{C}_S \\
&= T_f(A,\alpha)+T_{f'}(A,\alpha) \\
&= (T_f+T_{f'})(A,\alpha) && \text{by definition 2.7} \\
&= [T_{cat}(f)+T_{cat}(f')]\ (A,\alpha)
\end{aligned}$$

$\Longrightarrow T_{cat}(f+f') = T_{cat}(f) + T_{cat}(f')$

$\Longrightarrow T_{cat}$ is additive. Similarly, we can show that T^{cat} is also an additive functor.//

Definition 2.10. A covariant functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is said to be <u>left limit preserving</u> if whenever $\{A \xrightarrow{\alpha_i} A_i\}$ is a limit of a diagram D over the scheme Σ in $\mathcal{C}$, then $\{F(A) \xrightarrow{F(\alpha_i)} F(A_i)\}$ is limit of the diagram FD over the scheme Σ in $\mathcal{C}'$ [13] .

Similarly , we can define right limit preserving functor.

Now , we shall show that T_{cat} is a left limit preserving functor .

Proposition 2.3. The functor T_{cat} is left limit preserving.

Proof. Let $\{L \xrightarrow{\alpha_i} D_i\}_{i\in I}$ be a left limit of a diagram D over some scheme $\Sigma = (I,M,d)$ over the category $\mathcal{C}$. Then we have to prove that $\{T_{cat}(L) \xrightarrow{T_{cat}(\alpha_i)} T_{cat}(D_i)\}_{i\in I}$ is the left limit of $T_{cat} \circ D$ over the same scheme in $\mathcal{C}at$.

Let $m \in M$ be such that $d(m) = (i,j)$. Then consider the diagram

$$\begin{tikzcd}
& T_{cat}(D_i) \arrow[d, "T_{cat}(m)"] \\
T_{cat}(L) \arrow[ur, "T_{cat}(\alpha_i)"] \arrow[r, "T_{cat}(\alpha_j)"'] & T_{cat}(D_j)
\end{tikzcd}$$

By covariance of T_{cat} ,

$T_{cat}(m)\ T_{cat}(\alpha_i) = T_{cat}(m\ \alpha_i) = T_{cat}\ (\alpha_j)$ since

$\{L \longrightarrow D_i\}_{i \in I}$ is compatible family.

Hence above diagram is commutative , for all $m \in M$.

$\Longrightarrow$ that the family $\{T_{cat}(L) \longrightarrow T_{cat}(D_i)\}_{i \in I}$ is compatibles.

Next , consider a category $\mathcal{B}$ in $\mathcal{C}at$ with a family $\{\mathcal{B} \xrightarrow{\beta_i} T_{cat}(D_i)\}_{i \in I}$ of functors such that the following diagram is commutative , for all $m \in M$.

$$\begin{tikzcd}
 & T_{cat}(D_i) \arrow[d, "T_{cat}(m)"] \\
\mathcal{B} \arrow[ur, "\beta_i"] \arrow[r, "\beta_j"'] & T_{cat}(D_j)
\end{tikzcd}$$

i.e. $T_{cat}(m)\beta_i = \beta_j$, $m \in M$.

Let B be an object of $\mathcal{B}$. Then $\beta_i(B) \in T_{cat}(D_i) = \mathcal{C}_{D_i}$.

$\Longrightarrow$ there exists an object $Q_{B,i} \in \mathcal{C}$ (say) and a morphism $\gamma_{B,i} : Q_{B,i} \longrightarrow D_i$ such that $\beta_i(B) = (Q_{B,i}, \gamma_{B,i}) \in \mathcal{C}_{D_i}$.

Similarly , $\beta_j(B) = (Q_{B,j}, \gamma_{B,j})$. But by compatibility ,

$T_{cat}(m)\ \beta_i = \beta_j \Longrightarrow$

$$T_{cat}(m)\beta_i(B) = \beta_j(B) \Longrightarrow T_{cat}(m)\ (Q_{B,i}, \gamma_{B,i}) = (Q_{B,j}, \gamma_{B,j})$$

$$\Longrightarrow (Q_{B,i}, m\gamma_{B,i}) = (Q_{B,j}, \gamma_{B,j})$$

$$\Longrightarrow Q_{B,i} = Q_{B,j} = Q_B(\text{say}),\ \forall\ i \text{ and } j$$

$$\text{and } m\ \gamma_{B,i} = \gamma_{B,j}\quad ,\ \forall\, m \in M.$$

i.e. we have a compatible family $\left\{Q_B \xrightarrow{\gamma_{B,i}} D_i\right\}_{i \in I}$ of morphsims in $\mathcal{C}$. But , as $\left\{L \xrightarrow{\alpha_i} D_i\right\}_{i \in I}$ is left limit of D , there exists a unique morphism $\gamma_B : Q_B \longrightarrow L$ such that the following triangles

$$\begin{tikzcd} & L \arrow[d, "\alpha_i"] \\ Q_B \arrow[ur, "\gamma_B"] \arrow[r, "\gamma_{B,i}"'] & D_i \end{tikzcd} \qquad \ldots(2.d)$$

are commutative for all i .

Next , let $b : B \longrightarrow B'$ be a morphism in $\mathcal{B}$. Then, by compatibility, $T_{cat}(m)\beta_i(b) = \beta_j(b)$, but, by definition of T_{cat},

$T_{cat}(m)\ \beta_i(b) = \beta_i(b) : Q_{B,i} \longrightarrow Q_{B',i}$, such that the following diagrams

$$\begin{tikzcd}
Q_{B,i} \arrow[dd, "\beta_i(b)"'] \arrow[dr, "\gamma_{B,i}"] \arrow[drr, "m\gamma_{B,i}"] & & \\
& D_i \arrow[r, "m"] & D_j \\
Q_{B',i} \arrow[ur, "\gamma_{B',i}"'] \arrow[urr, "m\gamma_{B',i}"'] & &
\end{tikzcd}$$

are commutative for all i.

But $\quad Q_{B,i} = Q_{B,j} \qquad m\gamma_{B,i} = \gamma_{B,j}$

and

$$Q_{B',i} = Q_{B',j} \qquad m\gamma_{B',i} = \gamma_{B',j}$$

therefore , we have the following commutative diagrams for all j

$$\begin{tikzcd}
Q_{B,j} \arrow[dd, "\beta_j(b)"'] \arrow[dr, "\gamma_{B,j}"] & \\
& D_j \\
Q_{B',j} \arrow[ur, "\gamma_{B',j}"'] &
\end{tikzcd}$$

$\Longrightarrow \beta_i(b) = \beta_j(b) = \xi_b$ (say) for all i and j such that the following diagrams

$$\begin{tikzcd}
Q_{B,i} \arrow[dr, "\gamma_B"] \arrow[drr, "\gamma_{B,j}"] \arrow[dd, "\xi_b"'] & & \\
& L \arrow[r, dashed, "\alpha_i"] & D_j \\
Q_{B',i} \arrow[ur, "\gamma_{B'}"] \arrow[urr, "\gamma_{B',j}"'] & &
\end{tikzcd}$$

are commutative for all i and j.

Now , define a twofold map $\psi : \mathcal{B} \longrightarrow T_{cat}(L) = \mathcal{C}_L$ such that

(i) $\psi(B) = (Q_B, \gamma_B)$, $\forall$ all objects $B \in \mathcal{B}$

and (ii) $\psi(b) = \xi_b$, $\forall$ morphisms $b \in \mathcal{B}$.

This twofold map ψ satisfies the following conditions :

(i) $\psi(I_B) = \beta_i(I_B) = I_{\beta_i(B)} = I_{\psi(B)}$

(ii) let $b : B \longrightarrow B'$ and $b' : B' \longrightarrow B''$ be two morphisms in $\mathcal{B}$. Then

$$\psi(b'b) = \xi_{b'b} = \beta_i(b'b) = \beta_i(b')\beta_i(b) = \xi_{b'}\ \xi_b$$

$$= \psi(b')\ \psi(b)$$

and hence , ψ is a covariant functor from $\mathcal{B} \longrightarrow \mathcal{C}_L$

Next, let $B \in \mathcal{B}$

$$\left[T_{cat}(\alpha_i)\psi\right](B) = T_{cat}(\alpha_i)\ \psi\ (B)$$

$$= T_{\alpha_i}\ (Q_B, \Upsilon_B\ )$$

$$= (Q_B, \alpha_i \Upsilon_B) \qquad \text{by definition of } T_{\alpha_i}$$

$$= (Q_B, \Upsilon_{B,i}) \qquad \text{by compatibility}$$

$$= \beta_i(B) \qquad \text{by definition}$$

and for any morphism $b \in \mathcal{B}$

$$\left[T_{cat}(\alpha_i)\psi\right](b) = T_{cat}(\alpha_i)\psi(b) = T_{cat}(\alpha_i)\ \xi_b$$

$$= T_{\alpha_i}\ (\xi_b) = \xi_b = \beta_i(b)$$

$$\Longrightarrow T_{cat}(\alpha_i)\psi = \beta_i\ ,\quad \forall\ i \in I.$$

Thus, ψ exists which satisfies the requirements of commutativity of the diagram (2.a).

For uniqueness of ψ , let $\psi' : \mathcal{B} \longrightarrow \mathcal{C}_L$ be another covariant functor such that the following diagrams

$$\begin{tikzcd}
 & \mathcal{C}_L \arrow[d, "T_{cat}(\alpha_i)"] \\
\mathcal{B} \arrow[ur, "\psi'"] \arrow[r, "\beta_i"'] & \mathcal{D}_i
\end{tikzcd}$$

are commutative for all i.

$\Longrightarrow T_{cat}(\alpha_i)\psi' = T_{cat}(\alpha_i)\psi$

$\Longrightarrow T_{cat}(\alpha_i)\psi'(B) = T_{cat}(\alpha_i)\psi(B), \ \forall B \in \mathcal{B}$

$\Longrightarrow T_{cat}(\alpha_i)(Q'_B, \gamma'_B) = T_{cat}(\alpha_i)(Q_B, \gamma_B)$

$\Longrightarrow (Q'_B, \alpha_i\gamma'_B) = (Q_B, \alpha_i\gamma_B)$

$\Longrightarrow Q'_B = Q_B$ and $\alpha_i\gamma'_B = \alpha_i\gamma_B$

but , γ_B is unique , $\forall$ B such that diagram (2.d) is commutative $\Longrightarrow \gamma_B = \gamma'_B$

$\Longrightarrow (Q'_B, \gamma'_B) = (Q_B, \gamma_B)$

$\Longrightarrow \psi'(B) = \psi(B)$, $\quad \forall B \in \mathcal{B}$.

Also , we have $\psi'(b) = \psi(b)$, $\forall\, b : B \longrightarrow B'$ in $\mathcal{B}$

$\Longrightarrow \psi = \psi' \Longrightarrow \psi$ is unique. ||

Proposition 2.4. The functor $T^{cat} : \mathcal{C} \longrightarrow \mathcal{C}at$ takes right limit of a diagram D over a scheme $\Sigma = (I,M,d)$ in the category $\mathcal{C}$ to the left limit of the composite functor $T^{cat} \circ D$ over the same scheme in $\mathcal{C}at$.

Proof. Let $\{D_i \xrightarrow{\alpha_i} R\}_{i \in I}$ be a right limit of D over the scheme Σ in $\mathcal{C}$. Then we have to prove that $\{T^{cat}(R) \longrightarrow T^{cat}(D_i)\}_{i \in I}$ is the left limit of $T^{cat} \circ D$.

(i) Since the family $\{D_i \xrightarrow{\alpha_i} R\}_{i \in I}$ is cocompatible and T^{cat} is a contravariant functor,

$$\alpha_i m = \alpha_j \Longrightarrow T^{cat}(\alpha_i m) = T^{cat}(\alpha_j)$$
$$\Longrightarrow T^{cat}(m)\; T^{cat}(\alpha_i) = T^{cat}(\alpha_j)$$

$\Longrightarrow$ that the family $\{T^{cat}(R) \longrightarrow T^{cat}(D_i)\}_{i \in I}$ is compatible.

(ii) Next , we consider another compatible family $\{\mathcal{B} \xrightarrow{\beta_i} T^{cat}(D_i)\}_{i \in I}$ of morphisms (contravariant functors) in $\mathcal{C}at$.

Let $B \in \mathcal{C} \implies \beta_i(B) \in T^{cat}(D_i) = \mathcal{C}^{D_i} \implies$ there exist an object $Q_{B,i} \in \mathcal{C}$ and a morphism $\gamma_{B,i} : D_i \longrightarrow Q_{B,i} \in \mathcal{C}$ such that $\beta_i(B) = (\gamma_{B,i}, Q_{B,i})$ and similarly, $\beta_j(B) = (\gamma_{B,j}, Q_{B,j})$. But, compatibility of the considered family implies that $T^{cat}(m)\ \beta_i(B) = \beta_j(B)$

$$\implies (\gamma_{B,i}m, Q_{B,i}) = (\gamma_{B,j}, Q_{B,j})$$

$$\implies Q_{B,i} = Q_{B,j} = Q_B \text{(say) and } \gamma_{B,i}m = \gamma_{B,j},\ \forall\, i,j \in I$$

$\implies$ that we have a cocompatible family $\{D_i \xrightarrow{\gamma_{B,i}} Q_B\}_{i\in I}$ of morphisms in $\mathcal{C}$ $\implies$ there exists a unique morphism $\gamma_B : R \longrightarrow Q_B$ such that the following diagram is commutative

$$\begin{tikzcd} D_i \arrow[r, "\alpha_i"] \arrow[dr, "\gamma_{B,i}"'] & R \arrow[d, "\gamma_B"] \\ & Q_B \end{tikzcd}$$

for all i.

Next, if we have a morphism $b : B \longrightarrow B'$ in $\mathcal{C}$, then

$\beta_i(b) = \beta_j(b) = \xi_b$ (say) for all i and j.

We define,now, a twofold map

$$\psi : \mathcal{B} \longrightarrow T^{cat}(L)$$

such that

(i) $\psi(B) = (\Upsilon_B, Q_B)$, $\forall\, B \in \mathcal{B}$

and

(ii) $\psi(b) = \xi_b$, $\forall\, b \in \mathcal{B}$

Obviously , ψ is a contravariant functor and

$$[T^{cat}(\alpha_i)\psi]\ (B) = T^{cat}(\alpha_i)\psi(B) = T^{\alpha_i}(\Upsilon_B, Q_B)$$

$$= (\Upsilon_B\alpha_i, Q_B) = (\Upsilon_{B,i}, Q_{B,i}) = \beta_i(B)$$

and $[T^{cat}(\alpha_i)\psi]\ (b) = T^{cat}(\alpha_i)\psi(b) = T^{cat}(\alpha_i)\beta_i(b) = \beta_i(b)$

$\Longrightarrow T^{cat}(\alpha_i)\psi = \beta_i$, $\forall\, i \in I$.

For the uniqueness of ψ , applying the definition of Υ_B, proceed as for the case of uniqueness of ψ in proposition 2.3 above.

In [13] , we have the following proposition :

If a category $\mathcal{C}$ has products then a covariant functor $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is limit preserving if and only if it preserve finite products and intersections.

This proposition and the above propositions 2.3 and 2.4 imply the following corollaries :

Corollary 2.1. Let $\mathcal{C}$ be a category with products. Then T_{cat} preserves products , finite intersections and equalizers.

Corollary 2.2. Let $\mathcal{C}$ be a category with coproducts. Then T^{cat} carries coproducts , finite cointersections and coequalizers to products , finite intersections and equalizers respectively.

2.4. Functors on $\mathcal{C}_S$ and $\mathcal{C}^S$ induced by a given functor

We subdivide this section in two parts. In first part 2.4.1 , we define and study the functor $T_S : \mathcal{C}_S \longrightarrow \mathcal{C}'_{S'}$, when the given functor $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is covariant.functor. In the second part 2.4.2, we define and study $T_S: \mathcal{C}_S \longrightarrow \mathcal{C}'^{S'}$, where $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is a contravariant functor. For the covariant case , it is shown that (i) If T is faithful ,then T_S is faithful (ii) If T is full and faithful both, then T_S is full (iii) If T is an embedding , then T_S is also an embedding (iv) If T is exact,then T_S is exact (v) If S is a terminal object and T is limit preserving then T_S is also limit preserving (vi) If T has an adjoint, then T_S has also an adjoint (vii) T_S commutes with T_S. Similar type of results are proved in 2.4.2 also.

2.4.1. When given functor is covariant

Let $\mathcal{C}$ be a category and S be an object of $\mathcal{C}$ and $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a covariant functor. Then, we define functors $T_S : \mathcal{C}_S \longrightarrow \mathcal{C}'_{S'}$ and study its preservation properties.

Definition 2.11. Let $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a covariant functor and S be an object of $\mathcal{C}$. Define a twofold map

$$T_S : \mathcal{C}_S \longrightarrow \mathcal{C}'_{S'} \quad , \quad \text{where } S' = T(S) \in \mathcal{C}'$$

such that

(i) $T_S(A,\alpha) = (T(A),T(\alpha)), \quad \forall$ objects $(A,\alpha) \in \mathcal{C}_S$

and

(ii) for every $m : (A,\alpha) \longrightarrow (B,\beta)$ in $\mathcal{C}_S$,

$$T_S(m) = T(m) : (T(A),T(\alpha)) \longrightarrow (T(B),T(\beta))$$

The map is well defined as $T(m) \in \mathcal{C}_{T(S)}$ since the following diagram

$$\begin{tikzcd} T(A) \arrow[rr, "T(m)"] \arrow[dr, "T(\alpha)"'] & & T(B) \arrow[dl, "T(\beta)"] \\ & T(S) & \end{tikzcd}$$

is commutative.

Also ,

$$T_S(I_{(A,\alpha)}) = T_{I(A)} = I_{T(A)} = I_{(T(A)\ T(\alpha))} = I_{T_S(A,\alpha)}$$

and

$$T_S(m_1m_2) = T(m_1m_2) = T(m_1)\ T(m_2) = T_S(m_1)\ T_S(m_2).$$

Thus , T_S is a covariant functor.

Now , we investigate the properties of T induced in T_S.

Proposition 2.5. If T is faithful , then T_S is also faithful.

Proof. Let $m_1 \neq m_2$ be two morphisms in $\mathcal{C}_S \Longrightarrow m_1 \neq m_2$ in $\mathcal{C}$. Since T is faithful , $T(m_1) \neq T(m_2) \Longrightarrow T_S(m_1) \neq T_S(m_2)$, by definition.

If T is full we cannot say that T_S is full but, it is full if we assume that T is full and faithful both. This is the content of the following lemma :

Proposition 2.6. If T is full and faithful functor, then T_S is full.

Proof. We have show that the mapping

$$[(A,\alpha),(B,\beta)] \longmapsto [T_S(A,\alpha),T_S(B,\beta)] = [(T(A),T(\alpha)),(T(B),T(\beta))]$$

is onto.

Let $\xi \in [(T(A),T(\alpha)) , (T(B),T(\beta))] \quad \mathcal{C}'_{S'}$

$\Longrightarrow \quad \xi : T(A) \longrightarrow T(B)$ in $\mathcal{C}'$ such that the diagram

$$\begin{tikzcd}
T(A) \arrow[rr, "\xi"] \arrow[dr, "T(\alpha)"'] & & T(B) \arrow[dl, "T(\beta)"] \\
& T(S) &
\end{tikzcd}$$

is commutative.

Since T is full, there exists a morphism $\phi : A \longrightarrow B$ in $\mathcal{C}$ such that $T(\phi) = \xi \Longrightarrow T(\alpha) = T(\beta)\xi = T(\beta)T(\phi)=T(\beta\phi)$. Now , as T is faithful , we have

$$\beta\phi = \alpha$$

$$\Longrightarrow \phi \in \mathcal{C}_S \quad , \text{ such that } T_S(\phi) = \xi$$

$\Longrightarrow T_S$ is full.

Remark 2.1. If T_S is full , faithful , we cannot say anything about T.

Proposition 2.7. If T is an embedding from a category $\mathcal{C}$ to a category $\mathcal{C}'$, then $T_S : \mathcal{C}_S \longrightarrow \mathcal{C}'_{S'}$, where $S'=T(S)$, is also an embedding.

Proof. T_S is faithful , by proposition 2.5 , hence , to show that T_S is an embedding , we have to show that it preserves distinct objects. For this , let $(A,\alpha) \neq (B,\beta)$ in $\mathcal{C}_S$

$$\Longrightarrow \text{either } A \neq B \text{ or } \alpha \neq \beta$$

if $A \neq B$, then $T(A) \neq T(B)$ because , T is an embedding

$\Longrightarrow (T(A),T(\alpha) \neq (T(B),T(\beta))$ i.e. $T_S\ (A,\alpha) \neq T_S\ (B,\beta)$.

However , if $\alpha \neq \beta$ then $T(\alpha) \neq T(\beta)$, for T is faithful

$\Longrightarrow (T(A),T(\alpha) \neq (T(B),T(\beta))$ i.e. $T_S(A,\alpha) \neq T_S(B,\beta)$

$\Longrightarrow T_S$ is an embedding.

Proposition 2.8. If $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is a covariant exact functor , then $T_S : \mathcal{C}_S \longrightarrow \mathcal{C}'_{S'}$, $S' = T(S)$, is also an exact covariant functor.

Proof. To show the preservation of exactness , it is sufficient to show that if $A \xrightarrow{f} B \xrightarrow{g} C$ is exact in $\mathcal{C}$ then $(A,\alpha) \xrightarrow{f} (B,\beta) \xrightarrow{g} (C,\gamma)$ is exact in $\mathcal{C}_S$. For this, let $((D,\delta),i)$ be the image of f contained in (B,β) in $\mathcal{C}_S$. Then we have to prove that $(D,\delta) \xrightarrow{i} (B,\beta)$ is kernel of g in $\mathcal{C}_S$.

$$(D,\delta) \xrightarrow{i} (B,\beta) \xrightarrow{g} (C,\gamma) \;=\; \begin{tikzcd} D \arrow[r, "i"] \arrow[dr, "\delta"'] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[dl, "\gamma"] \\ & S & \end{tikzcd}$$

Since $D = \mathrm{Im} f = \mathrm{Ker}\, g$ in $\mathscr{C}$, $gi = o$ in $\mathscr{C}$ and hence in $\mathscr{C}_S$.
Next, if we have $(D',\delta') \xrightarrow{i'} (B,\beta) \xrightarrow{g} (C,\gamma)$ such that $gi' = o$ in $\mathscr{C}_S \Longrightarrow gi' = o$ in $\mathscr{C} \Longrightarrow$ there exists a unique morphism $\xi : D' \longrightarrow D$ such that $D' \xrightarrow{\xi} D \xrightarrow{i} B = D' \xrightarrow{i'} B$.
Now $\delta\xi = \beta i \xi = \beta i' = \delta'$

$\Longrightarrow \xi : (D',\delta') \longrightarrow (D,\delta)$ belongs to $\mathscr{C}_S$ such that the following diagram

$$\begin{tikzcd} (D',\delta') \arrow[dr, "i'"] \arrow[d, "\xi"'] & \\ (D,\delta) \arrow[r, "i"'] & (B,\beta) \end{tikzcd}$$ is commutative

$\Longrightarrow ((D,\delta),i) = \mathrm{Ker}\, g$ in $\mathscr{C}_S$.

Proposition 2.9. If $T : \mathscr{C} \longrightarrow \mathscr{C}'$ is a left limit preserving covariant functor and S is a terminal object of $\mathscr{C}$, then $T_S : \mathscr{C}_S \longrightarrow \mathscr{C}'_{S'}$, $S' = T(S)$, is also a left limit preserving covariant functor.

In order to prove this proposition , we shall use the following lemma :

Lemma 2.4. Let S be a terminal object of $\mathcal{C}$. Then the family $\{(A,\alpha) \xrightarrow{\alpha_i} (D_i,d_i)\}_{i\in I}$ is a left limit of a diagram D over a scheme $\Sigma = (I,M,d)$ in $\mathcal{C}_S$ if and only if $\{A \xrightarrow{\alpha_i} D_i\}_{i\in I}$ is left limit of D over the same scheme in $\mathcal{C}$.

Proof. Let $\{(A,\alpha) \xrightarrow{\alpha_i} (D_i,d_i)\}_{i\in I}$ be left limit of D over the scheme Σ in $\mathcal{C}_S$. Then $m\alpha_i = \alpha_j$ in $\mathcal{C}_S$ and hence in $\mathcal{C}$. This implies that the family $\{A \xrightarrow{\alpha_i} D_i\}_{i\in I}$ is compatible in $\mathcal{C}$.

Next , let $\{B \xrightarrow{\beta_i} D_i\}_{i\in I}$ be another compatible family of morphisms in $\mathcal{C}$. Since S is a terminal object , $d_i\beta_i = d_j\beta_j = \beta$ (say) for all i and j . Thus, we have a family $\{(B,\beta) \xrightarrow{\beta_i} (D_i,d_i)\}_{i\in I}$ of morphisms in $\mathcal{C}_S$, but as $\{(A,\alpha) \xrightarrow{\alpha_i} (D_i,d_i)\}_{i\in I}$ is left limit of D , there exists a unique morphism $\gamma : (B,\beta) \longrightarrow (A,\alpha)$ in $\mathcal{C}_S$ such that

$$(B,\beta) \xrightarrow{\gamma} (A,\alpha) \xrightarrow{\alpha_i} (D_i,d_i) = (B,\beta) \xrightarrow{\beta_i} (D_i,d_i) \ , \ \forall i \ .$$

Hence, $\gamma : B \longrightarrow A$ in $\mathcal{C}$ is such that

$B \xrightarrow{\gamma} A \xrightarrow{\alpha_i} D_i = B \xrightarrow{\beta_i} D_i$, $\forall i$.

Conversely , let $\{A \xrightarrow{\alpha_i} D_i\}_{i\in I}$ be the limit of the diagram D over the scheme Σ in $\mathcal{C}$. Then $m\alpha_i = \alpha_j$ in $\mathcal{C}$ and hence in $\mathcal{C}_S$. This implies that the family $\{(A,\alpha) \xrightarrow{\alpha_i} (D_i,d_i)\}_{i\in I}$ is compatible.

Next , if $\{(B,\beta) \xrightarrow{\beta_i} (D_i,d_i)\}_{i\in I}$ is another family of morphisms in $\mathcal{C}_S$, then $\{B \xrightarrow{\beta_i} D_i\}_{i\in I}$ is another family of morphisms in $\mathcal{C}$. Therefore, there exists a unique morphism $\gamma : B \longrightarrow A$ such that $B \xrightarrow{\gamma} A \xrightarrow{\alpha_i} D_i = B \xrightarrow{\beta_i} D_i$, $\forall i$, Now , as S is terminal object $\beta = \alpha\gamma \Longrightarrow \gamma \in \mathcal{C}_S$ and $(B,\beta) \xrightarrow{\gamma} (A,\alpha) \xrightarrow{\alpha_i} (D_i,d_i) = (B,\beta) \xrightarrow{\beta_i} (D_i,d_i)$, $i\in I$.

Proof of the proposition 2.9. Let $\{(A,\alpha) \xrightarrow{\alpha_i} (D_i,d_i)\}_{i\in I}$ be a lefttlimit of a diagram D over the scheme Σ in $\mathcal{C}_S$.

To prove the proposition, we have to prove that

$\{T_S(A,\alpha) \longrightarrow T_S(D_i,d_i)\}_{i\in I}$ is left limit of $T_S \circ D$ over Σ in $\mathcal{C}'_{S'}$.

Now , $T_S(m)\, T_S(\alpha_i) = T_S(m\alpha_i) = T_S(\alpha_j)$. Hence the family is compatible.

Next , let $\left\{(B,\beta) \xrightarrow{\beta_i} T_S(D_i,d_i)\right\}_{i\in I}$ be another compatible family of morphisms in $\mathcal{C}'_{S'}$, that is , the following diagram is commutative , for all i and j :

$$\begin{tikzcd} & T_S(D_i,d_i) \arrow[dd, "T_S(m)"] \\ (B,\beta) \arrow[ur, "\beta_i"] \arrow[dr, "\beta_j"'] & \\ & T_S(D_j,d_j) \end{tikzcd}$$

Thus , the following diagram is commutative in $\mathcal{C}$ for all i and j

$$\begin{tikzcd} & T(D_i) \arrow[dd, "T(m)"] \\ B \arrow[ur, "\beta_i"] \arrow[dr, "\beta_j"'] & \\ & T(D_j) \end{tikzcd}$$

But , by lemma 2.4, $\{A \longrightarrow D_i\}_{i\in I}$ is left limit of D over the scheme Σ in $\mathcal{C}$ and since T is left limit preserving ,

$\left\{T(A) \xrightarrow{T(\alpha_i)} T(D_i)\right\}_{i\in I}$ is left limit of T o D in $\mathcal{C}'$ ==>

there exists a unique morphism γ : B ----> T(A) such that

$$B \xrightarrow{\gamma} T(A) \xrightarrow{T(\alpha_i)} T(D_i) = B \xrightarrow{\beta_i} T(D_i) \ , \ \forall\, i \in I.$$

Now ,

$$T(\alpha)\gamma = T(d_i\alpha_i)\gamma = T(d_i)\ T(\alpha_i)\gamma$$

$$= T(d_i)\ \beta_i = \beta$$

$\Longrightarrow \gamma : (B,\beta) \longrightarrow (T(A),T(\alpha)) = T_S(A,\alpha)$ belongs to $\mathcal{C}'_{S'}$, such that

$$(B,\beta) \xrightarrow{\gamma} T_S(A,\alpha) \xrightarrow{T_S(\alpha_i)} T_S(D_i,d_i) = (B,\beta) \xrightarrow{\beta_i} T_S(D_i,d_i)$$

$\Longrightarrow T_S$ is left limit preserving.

Proposition 2.10. If $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is a right limit preserving covariant functors and S is a terminal object of $\mathcal{C}$, then $T_S : \mathcal{C}_S \longrightarrow \mathcal{C}'_{S'}$, $S' = T(S)$ is also a right limit preserving functor.

In order to prove the above proposition , we shall similarly use the following lemma :

Lemma 2.5. Let S be a terminal object of a category $\mathcal{C}$ .Then the family $\left\{(D_i,d_i) \xrightarrow{\alpha_i} (A,\alpha)\right\}_{i \in I}$ is a right limit of of a diagram D over a scheme $\Sigma = (I,M,d)$ in $\mathcal{C}_S$ if and only if the family $\left\{D_i \xrightarrow{\alpha_i} A\right\}_{i \in I}$ is right limit of D over the same scheme in $\mathcal{C}$.

Proof of the lemma. Let $\{(D_i,d_i) \xrightarrow{\alpha_i} (A,\alpha)\}_{i\in I}$ be the right limit of the diagram D over the scheme Σ in $\mathcal{C}_S$. Then $\alpha_i m = \alpha_j$ in $\mathcal{C}_S$ and hence in $\mathcal{C}$ ==> that the family $\{D_i \xrightarrow{\alpha_i} A\}_{i\in I}$ is cocompatible in $\mathcal{C}$.

Next, let $\{D_i \xrightarrow{\beta_i} B\}_{i\in I}$ be another cocompatible family of morphism in $\mathcal{C}$. Since S is a terminal object, there exists a unique morphism $\beta : B \longrightarrow S$ such that $\beta\beta_i = d_i$, $\forall\, i \in I$. Thus, we have a family $\{(D_i,d_i) \xrightarrow{\beta_i} (B,\beta)\}_{i\in I}$ of morphism in $\mathcal{C}_S$. Now, $\{(D_i,d_i) \xrightarrow{\alpha_i} (A,d)\}_{i\in I}$ is right limit of D, there exists a unique morphism $\gamma : (A,\alpha) \longrightarrow (B,\beta)$ in $\mathcal{C}_S$ such that

$$(D_i,d_i) \xrightarrow{\alpha_i} (A,\alpha) \xrightarrow{\gamma} (B,\beta) = (D_i,d_i) \xrightarrow{\beta_i} (B,\beta),\ \forall\, i \in I .$$

Thus, there exists a unique morphism $\gamma : A \longrightarrow B$ in $\mathcal{C}$ such that

$$D_i \xrightarrow{\alpha_i} A \xrightarrow{\gamma} B = D_i \xrightarrow{\beta_i} B ,\ \forall\, i \in I .$$

Conversely, let $\{D_i \xrightarrow{\alpha_i} A\}_{i\in I}$ be a right limit of D over Σ in $\mathcal{C}$. Hence $\alpha_i m = \alpha_j$, $\forall\, i$, j in I and therefore in $\mathcal{C}_S$. Thus the family $\{(D_i,d_i)\xrightarrow{\alpha_i}(A,\alpha)\}_{i\in I}$ is cocompatible.

Next , if we have $\{(D_i,d_i)\xrightarrow{\beta_i}(B,\beta)\}_{i\in I}$ a cocompatible family of morphisms in $\mathcal{C}_S$, then $\{D_i\xrightarrow{\beta_i}B\}_{i\in I}$ is a cocompatible family of morphisms in $\mathcal{C}$ and hence ,there exists a unique morphism $\gamma : A \longrightarrow B$ in $\mathcal{C}$ such that

$$D_i \xrightarrow{\alpha_i} A \xrightarrow{\gamma} B = D_i \xrightarrow{\beta_i} B\ , \ \forall i \in I.$$

Now, as S is a terminal object , $\beta\gamma = \alpha \Longrightarrow \gamma \in \mathcal{C}_S$ such that

$$(D_i,d_i)\xrightarrow{\beta_i}(A,\alpha)\xrightarrow{\gamma}(B,\beta)=(D_i,d_i)\xrightarrow{\beta_i}(B,\beta),\ \ \forall i \in I$$

$\Longrightarrow \{(D_i,d_i)\xrightarrow{\alpha_i}(A,\alpha)\}_{i\in I}$ is right limit of D.

Proof of the proposition 2.10. Let $\{(D_i,d_i)\xrightarrow{\alpha_i}(A,\alpha)\}_{i\in I}$ be a right limit of a diagram D over the scheme Σ in $\mathcal{C}_S$. Then we have to prove that $\{T_S(D_i,d_i)\xrightarrow{T_S(\alpha_i)}T_S(A,\alpha)\}_{i\in I}$ is right limit of T_S o D over the scheme Σ in $\mathcal{C}'_{S'}$.

Now , as T_S is covariant functor

$$T_S(\alpha_i)\ T_S(m) = T_S(\alpha_i m) = T_S(\alpha_j),\ \forall i\ ,\ j \in I\ ,$$

$\Longrightarrow$ that the family $\{T_S(D_i,d_i) \xrightarrow{T_S(\alpha_i)} T_S(A,\alpha)\}_{i\in I}$ is cocompatible. Next, if $\{T_S(D_i,d_i) \xrightarrow{\beta_i} (B,\beta)\}_{i\in I}$ is another cocompatible family of morphisms in $\mathcal{C}'_{S'}$, then $\{T(D_i) \xrightarrow{\beta_i} B\}_{i\in I}$ is a cocompatible family of mrophsims in $\mathcal{C}$. And, by lemma 2.5, $\{D_i \xrightarrow{\alpha_i} A\}_{i\in I}$ is right limit of D over the scheme Σ in $\mathcal{C}$ and T is right limit preserving functor $\Longrightarrow$ the family $\{T(D_i) \xrightarrow{T(\alpha_i)} T(A)\}_{i\in I}$ is right limit of T o D over Σ in $\mathcal{C}'$ $\Longrightarrow$ there exists a unique morphism $\gamma : T(A) \longrightarrow B$ such that

$$T(D_i) \xrightarrow{T(\alpha_i)} T(A) \xrightarrow{\gamma} B = T(D_i) \xrightarrow{\beta_i} B, \ \forall i \in I .$$

Now, since

$$T(\alpha)T(\alpha_i) = T(d_i) = \beta\beta_i = \beta\gamma T(\alpha_i), \forall\, i\in I ,$$

and $T(\alpha_i)$'s are right limit morphisms,

$T(\alpha) = \beta\gamma \Longrightarrow \gamma : (T(A),T(\alpha)) \xrightarrow{\beta_i} (B,\beta)$ belongs to $\mathcal{C}'_{S'}$ such that

$$(T(D_i),T(d_i)) \xrightarrow{\alpha_i} (T(A),T(\alpha)) \xrightarrow{\gamma} (B,\beta)$$

$$= (T(D_i),T(d_i)) \xrightarrow{\beta_i} (B,\beta).$$

Hence, the result follows. $\|$

Proposition 2.11. Let $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a covariant functor , which has a left adjoint $T' : \mathcal{C}' \longrightarrow \mathcal{C}$, and S be an object of $\mathcal{C}$. Then $T_S : \mathcal{C}_S \longrightarrow \mathcal{C}'_{S'}$, $S' = T(S)$, also has a left adjoint.

Proof. First , we define a covariant functor $T'_{S'} : \mathcal{C}'_{S'} \longrightarrow \mathcal{C}_S$ such that

$$\text{(i)}\ T'_{S'}(A',\alpha') = (T'(A'), \psi_S\, T'(\alpha')),\ \forall \text{ objects } (A',\alpha') \in \mathcal{C}'_{S'}$$

where $\psi_S = \eta^{-1}_{T(S),S}(I_{T(S)}) : T'T(S) \longrightarrow S$, as we have explained in the section 0.2.5 , $T'_{S'}$ is well defined ,since for $A \in \mathcal{C}$, $A' \in \mathcal{C}'$

$$\eta_{A',A} : [T'(A'),A] \longrightarrow [A',\ T(A)]$$

is the natural equivalence for the adjointness of T and T' , and ,therefore , we have

$$\psi_S\, T'(\alpha') = T'(A') \xrightarrow{T'(\alpha')} T'T(S) \xrightarrow{\psi_S} S$$

$$\Longrightarrow (T'(A'),\ \psi_S\, T'(\alpha')) \in \mathcal{C}_S ,$$

Also , (ii) $T'_{S'}(m') = T'(m')$, $\quad m' : (A',\alpha') \longrightarrow (B',\beta')$ in $\mathcal{C}'_{S'}$.

Obviously, $T'_{S'} : \mathcal{C}'_{S'} \longrightarrow \mathcal{C}_S$ is a covariant functor.

Now , we shall prove that $T'_{S'}$ is left adjoint of T_S. For this , we define a mapping η for all $(A',\alpha') \in \mathcal{C}'_{S'}$ and for all $(A,\alpha) \in \mathcal{C}_S$ as

$$\eta_{(A',\alpha'),(A,\alpha)} : \left[T'_{S'}(A',\alpha'),(A,\alpha)\right] \dashrightarrow \left[(A',\beta'),T_S(A,\alpha)\right]$$

In other words,

$$\eta_{(A',\alpha'),(A,\alpha)} : \left[(T'(A'),\psi_S T'(\alpha')),(A,\alpha)\right] \dashrightarrow \left[(A',\alpha'),(T(A),T(\alpha))\right]$$

such that

$$\eta_{(A',\alpha')(A,\alpha)}(\xi) = \eta_{A',A}(\xi), \forall \xi \in \left[(T'(A'),\psi_S T'(\alpha')),(A,\alpha)\right]$$

Since $\eta_{A',A}$ is natural equivalence , $\eta_{(A',\alpha'),(A,\alpha)}$ is also a natural equivalence if we show that the mapping is defined in $\mathcal{C}'_{S'}$, i.e. if the following diagram is commutative :

$$\begin{tikzcd}
A' \arrow[rr, dashed, "\eta_{A'A}(\xi)"] \arrow[dr, "\alpha'"'] & & T(A) \arrow[dl, "T(\alpha)"] \\
& T(S) &
\end{tikzcd}$$

i.e. if the following diagram is commutative

$$\begin{tikzcd}
A' \arrow[r, "\eta_{A',A}(\xi)"] \arrow[d, "\alpha'"] & T(A) \arrow[d, "T(\alpha)"] \\
T(S) \arrow[r, "I_{T(S)}"] & T(S)
\end{tikzcd}$$

But , this square is commutative , because the following commutative square under η corresponds to the above square

$$\begin{tikzcd}
T'(A') \arrow[r, "\xi"] \arrow[d, "T'(\alpha')"] & A \arrow[d, "\alpha"] \\
T'T(S) \arrow[r, "\psi_S"] & S
\end{tikzcd}$$

Hence , $T'_{S'}$ is left adjoint to T_S .

We show now that T_S commutes with the functor T_f of Definition 2.5.

Proposition 2.12. Let $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a covariant functor and $f : S \longrightarrow R$ be a morphism in $\mathcal{C}$. Then the following diagram

$$\begin{tikzcd}
\mathcal{C}_S \arrow[r, "T_S"] \arrow[d, "T_f"'] & \mathcal{C}'_{S'} \arrow[d, "T_{f'}"] \\
\mathcal{C}_R \arrow[r, "T_R"] & \mathcal{C}'_{R'}
\end{tikzcd}
\quad \text{where} \quad
\begin{aligned} S' &= T(S) \\ f' &= T(f) \\ R' &= T(R) \end{aligned}$$

is commutative.

Proof. Let (A,α) be an object of $\mathcal{C}_S$. Then

$$\begin{aligned}
T_{f'}\, T_S(A,\alpha) &= T_{f'}(T(A),T(\alpha)) = (T(A), f'T(\alpha)) \\
&= (T(A),\ T(f)\ T(\alpha)) \\
&= (T(A),\ T(f\alpha)) \\
&= T_R\ (A,\ f\alpha) \\
&= T_R(T_f(A,\alpha))
\end{aligned}$$

Thus $T_{f'}T_S(A,\alpha) = T_R\ T_f(A,\alpha)$, $\quad (A,\alpha) \in \mathcal{C}_S$ ...(2.e)

and if $m : (A,\alpha) \longrightarrow (B,\beta)$ is a morphism in $\mathcal{C}_S$, then

$$T_{f'}T_S(m) = T(m) = T_R\ T_f(m) \qquad \text{...(2.f)}$$

Now, (2.e) and (2.f) $\Longrightarrow T_{f'}\, T_S = T_R\ T_f$.

Remark 2.2. We define ,similarly, an induced functor $T^S : \mathcal{C}^S \longrightarrow \mathcal{C}'^{S'}$ for a given covariant functor $T : \mathcal{C} \longrightarrow \mathcal{C}'$ where $S' = T(S)$. All the preservation properties discussed above for T_S can also be checked easily for T^S. Therefore , we only state the preservation properties without providing proof any of them.

Definition 2.11. Let $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a covariant functor and S be an object of $\mathcal{C}$. Define a twofold map

$$T^S : \mathcal{C}^S \longrightarrow \mathcal{C}'^{S'}$$

such that

$$\text{(i) } T^S(\alpha,A) = (T(\alpha),T(A)), \qquad (\alpha,A) \in \mathcal{C}^S$$

and

$$\text{(ii) } T^S(m) = T(m) ,$$

$$m : (\alpha,A) \longrightarrow (\beta,B) \in \mathcal{C}^S$$

Then , T^S is a covariant functor. And the properties are as follows :

Proposition 2.13. If T is faithful , faithful full, exact and embedding, then $T^S : \mathcal{C}^S \longrightarrow \mathcal{C}'^{S'}$ is also faithful, full, exact and embedding respectively.

Proposition 2.14. If S is an initial object of category and $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is a limit preserving covariant functor then $T^S : \mathcal{C}^S \longrightarrow \mathcal{C}'^{S'}$ is also limit preserving functor.

Proposition 2.15. If $T : \mathcal{C} \longrightarrow \mathcal{C}'$ has a left adjoint $T' : \mathcal{C}' \longrightarrow \mathcal{C}$, then $T^S : \mathcal{C}^S \longrightarrow \mathcal{C}'^{S'}$, $S' = T(S)$, has $T'^{S'} : \mathcal{C}'^{S'} \longrightarrow \mathcal{C}^S$ a left adjoint , where $T'^{S'} : \mathcal{C}'^{S'} \longrightarrow \mathcal{C}^S$ is defined as follows :

(i) $T'^{S'}(\alpha', A') = (T'(\alpha')\phi_S \, , \, T'(A'))$,

$$\forall (\alpha', A') \in \mathcal{C}'^{S'}$$

where

$$\phi_S = \eta_{S,T'(S)}(I_{T'(S)}) : S \longrightarrow TT'(S)$$

and

$$\eta_{A',A} : [T(A'),A] \longrightarrow [A', T(A)]$$

is a natural equivalence for the adjointness of T and T'. Also,

(ii) $T'^{S'}(m) = \dot{T}(m)$,

$$m : (A,\alpha) \longrightarrow (B,\beta) \in \mathcal{C}'^{S'} .$$

We have the following proposition as partial dual of proposition 2.12.

Proposition 2.16. Let $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a covariant functor and $f : S \longrightarrow R$ be a morphism in $\mathcal{C}$. Then the following diagram is commutative

$$\begin{tikzcd}
\mathcal{C}^{S} \arrow[r, "T^{S}"] \arrow[d, "T^{f}"] & \mathcal{C}'^{S'} \arrow[d, "T^{f'}"] \\
\mathcal{C}^{R} \arrow[r, "T^{R}"] & \mathcal{C}'^{R'}
\end{tikzcd}$$

where $S' = T(S)$, $R' = T(R)$, $f' = T(f)$ and T^{f} is a functor as given in Definition 2.6.

2.4.2. <u>When given functor T is contravariant</u>

For the contravariant case , let $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a contravariant functor and S be an object of $\mathcal{C}$, we define functors $T_S : \mathcal{C}_S \longrightarrow \mathcal{C}'^{S'}$, $S' = T(S)$, and $T^S : \mathcal{C}^S \longrightarrow \mathcal{C}'_{S'}$ in the following :

Definition 2.12. Let $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a contravariant functor and S be an object of $\mathcal{C}$. Define a twofold map

$T_S : \mathscr{C}_S \dashrightarrow \mathscr{C}'^{S'}$, $S' = T(S)$ such that

(i) $T(A,\alpha) = (T(\alpha), T(A))$,

$\forall$ objects (A,α) of $\mathscr{C}_S$.

and

(ii) $T_S(m) = T(m): (T(\beta),T(B)) \dashrightarrow (T(\alpha),T(A))$,

$m : (A,\alpha) \dashrightarrow (B,\beta)$ in $\mathscr{C}_S$.

This map T_S is obviously a contravariant functor.

Definition 2.13. Let $T : \mathscr{C} \dashrightarrow \mathscr{C}'$ be a contravariant functor and S be an object of $\mathscr{C}$. Define a twofold map $T^S : \mathscr{C}^S \dashrightarrow \mathscr{C}'^{S'}$, $S' = T(S)$, such that

(i) $T^S(\alpha,A) = (T(A), T(\alpha))$,

$\forall$ objects $(\alpha,A) \in \mathscr{C}^S$,

and

(ii) $T^S(m) = T(m) : (T(B),T(\beta)) \dashrightarrow (T(A),T(\alpha))$,

morphisms $m : (\alpha,A) \dashrightarrow (\beta,B)$ in $\mathscr{C}^S$.

This map T^S is also a contravariant functor.

The following are the preservation properties of the functors

$T_S : \mathcal{C}_S \longrightarrow \mathcal{C}'^{S'}$ and $T^S : \mathcal{C}^S \longrightarrow \mathcal{C}'_{S'}$,which can be checked easily as before with slight modifications in the proofs for the case of covariant functors.

Proposition 2.17. If T is faithful , faithful full, exact and embedding , then T_S and T^S are faithful, full ,exact and embeddings respectively.

Proposition 2.18. If $T : \mathcal{C} \longrightarrow \mathcal{C}'$, a contravariant functor , takes right lmits/left limits of a diagram D over the scheme $\Sigma = (I,M,d)$ to left limits/right limits over the same scheme in $\mathcal{C}'$, then $T_S : \mathcal{C}_S \longrightarrow \mathcal{C}'^{S'}$ also takes right/left limits of diagram D in $\mathcal{C}_S$ to the left/right limit, Whenever S is terminal/initial object.

Similar type of proposition can also be stated for T^S, as below :

Proposition 2.19. If $T : \mathcal{C} \longrightarrow \mathcal{C}'$, a contravariant functor , takes right/left limit of a diagram D over the scheme $\Sigma = (I,M,d)$ in $\mathcal{C}$ to left/right limit of T o D over the same scheme in $\mathcal{C}'$, then $T^S : \mathcal{C}^S \longrightarrow \mathcal{C}'_{S'}$, also takes right/left limits of a diagram over the same scheme in $\mathcal{C}^S$ to left/right limit of the composite diagram over the same scheme in $\mathcal{C}'_{S'}$, whenever S is an initial/terminal object.

2.5. Functors induced by a functor through a terminal or initial object

Let $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a covariant/contravariant functor and S' be a terminal/an initial object of $\mathcal{C}'$. Then we define functors $\hat{T}_S : \mathcal{C}_S \longrightarrow \mathcal{C}'_{S'}$ / $\hat{T}^S : \mathcal{C}^S \longrightarrow \mathcal{C}'^{S'}$ and study their properties, some of which are similar to those for T_S and T^S where S is any object in $\mathcal{C}$.

Definition 2.14. Let $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a covariant functor, S be an object of $\mathcal{C}$ and S' be a terminal object of $\mathcal{C}'$. Then we define a twofold map :

$$\hat{T}_S : \mathcal{C}_S \longrightarrow \mathcal{C}'_{S'} \quad \text{such that}$$

(i) $\hat{T}_S(A,\alpha) = (T(A),\gamma)$, where $\gamma : T(A) \longrightarrow S'$ is a unique morphism, for all $(A,\alpha) \in \mathcal{C}_S$

and

(ii) $\hat{T}_S(m) = T(m)$ morphism $m \in \mathcal{C}_S$.

Obviously, $\hat{T}_S$ is a covariant functor.

Definition 2.15. If S' is an initial object of $\mathcal{C}'$, then we define

$$\hat{T}_S : \mathcal{C}^S \longrightarrow \mathcal{C}'^{S'} \quad \text{such that}$$

(i) $\hat{T}^S(\alpha, A) = (\gamma, T(A))$, where $\gamma : S' \longrightarrow T(A)$ is a unique morphism , $\forall$ objects (α, A) in $\mathcal{C}^S$,

and

(ii) $\hat{T}^S(m) = T(m)$ $\forall$ morphisms $m \in \mathcal{C}^S$.

This is also a covariant functor.

The following propositions can be chekced easily , as we have checked earlier.

Proposition 2.20. If $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is faithful, faithful full, exact, embedding and limit preserving , then $\hat{T}_S$ and $\hat{T}^S$ are faithful, full, exact , embedding and limit preserving respectively.

Also propositions 2.12 and 2.16 can also be obtained for the case of $\hat{T}_S$ and $\hat{T}^S$ as follows :

Proposition 2.21. Let $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a covariant functors and $f : S \longrightarrow R$ be a morphism in $\mathcal{C}$ and S', R' be terminal objects in $\mathcal{C}'$ corresponding to S and R and $f' : S' \longrightarrow R'$ be unique morphism in $\mathcal{C}'$ then

(i) $\hat{T}_{f'} \hat{T}_S = \hat{T}_R T_f$

If the objects S', R' are initial in $\mathcal{C}'$, then

(ii) $T^{f'} \hat{T}^S = \hat{T}^R T^f$.

Proof. For (i), let (A,α) be any object in $\mathcal{C}_S$ and $m : (A,\alpha) \longrightarrow (B,\beta)$ be any morphism in $\mathcal{C}_S$. Then

$$T_{f'}\hat{T}_S(A,\alpha) = T_{f'}(T(A),\gamma)$$

$$= (T(A),f'\gamma), \quad \text{where } T(A) \xrightarrow{\gamma} S' \xrightarrow{f'} R'.$$

Also $\hat{T}_R T_f(A,\alpha) = \hat{T}_R(A,f\alpha) = (T(A),\gamma')$, where $\gamma' : T(A) \longrightarrow R'$.

Since R' is terminal object $f'\gamma = \gamma'$.

Hence $T_{f'} \hat{T}_S = \hat{T}_R T_f$.

We can , similarly , obtain (ii) making necessary changes for initial object ,

Remark 2.3. In the case of T_S and T^S , we were not able to get a natural transformations for the two induced functors , the reason was generality of object over which the categories were defined. Now , by putting restriction on S' as terminal or initial object we are able to obtain preposition 2.22, which gives a natural transformation.

Proposition 2.22. Let T and T' be two covariant functors from a category $\mathcal{C}$ to a category $\mathcal{C}'$ and $\eta : T \longrightarrow T'$ be a natural transformation and S be an object of $\mathcal{C}$, $T(S) = S'$ be a terminal object of $\mathcal{C}'$. Then, there exists a natural transformation $\eta_S : T_S \longrightarrow T'_S : \mathcal{C}_S \longrightarrow \mathcal{C}'_{S'}$ such that the following diagram

$$\begin{tikzcd}
\hat{T}_S(A,\alpha) \arrow[r, "\hat{T}_S(m)"] \arrow[d, "\eta_S(A,\alpha)"'] & \hat{T}_S(B,\beta) \arrow[d, "\eta_S(B,\beta)"] \\
\hat{T}'_S(A,\alpha) \arrow[r, "\hat{T}'_S(m)"] & \hat{T}'_S(B,\beta)
\end{tikzcd}$$

is commutative.

Proof. Define $\eta_S : \hat{T}_S \longrightarrow \hat{T}'_S$ by the rule

$$\eta_S(A,\alpha) = \eta_A : T(A) \longrightarrow T'(A)$$

Thus, $\eta_S(A,\alpha)$ is a mapping from $\hat{T}_S(A,\alpha) \longrightarrow \hat{T}'_S(A,\alpha)$ where

$$\begin{tikzcd}
T(A) \arrow[rr, "\eta(A)"] \arrow[dr, "\alpha"'] & & T'(A) \arrow[dl, "\beta"] \\
& S' &
\end{tikzcd}$$

is commutative as $S' = T(S)$ is a terminal object

$\Longrightarrow \eta_S(A,\alpha) \in \mathcal{C}'_{S'}$. Thus the mapping is well defined.

Also,

$$\eta_S(B,\beta)\,\hat{T}_S(m) = \eta_S(B,\beta)T(m) = \eta(B)\,T(m)$$

$$= T'(m)\,\eta(A) \quad \text{as } \eta \text{ is a natural transformation}$$

$$= \hat{T}'_S(m)\,\eta_S(A,\alpha)$$

This proves that $\mathcal{C}_S$ is a natural transformation from $\hat{T}_S$ to $\hat{T}'_S$. ||

Now, for the contravariant case, let $\hat{T} : \mathcal{C} \longrightarrow \mathcal{C}'$ be a contravariant functor, where $S \in \mathcal{C}$ and $S' \in \mathcal{C}'$ and $S' = T(S)$ be an initial object. Then, we define contravariant functors $\hat{T}_S : \mathcal{C}_S \longrightarrow \mathcal{C}'_{S'}$ and $\hat{T}^S : \mathcal{C}^S \longrightarrow \mathcal{C}'_{S'}$ as follows :

Definition 2.16. Define twofold map $\hat{T}_S : \mathcal{C}_S \longrightarrow \mathcal{C}'_{S'}$ such that

(i) $\hat{T}_S(A,\alpha) = (T(A),\Upsilon)$ objects $(A,\alpha) \in \mathcal{C}_S$

and

(ii) $\hat{T}_S(m) = T(m)$.

Obviously, $\hat{T}_S$ is a contravariant functor.

Definition 2.17. Define twofold map $\hat{T}^S : \mathcal{C}^S \longrightarrow \mathcal{C}_{S'}$ such that

(i) $\hat{T}^S(\alpha, A) = (T(A), \Upsilon)$

and

(ii) $\hat{T}^S(m) = T(m)$.

This is also a contravariant functor.

These functors carry over all the dual properties to those proved for the covariant case.

<u>References</u>

Bucur [1] , Mitchell [13] .

CHAPTER III

FIBERED STRUCTURES

3.0. Introduction. We have already defined projectivity, injectivity , reflection , generators and cogenerators etc. in chapter 0 (preliminaries). In this chapter , we define fibered/cofibered projectivity , injectivity , reflection and fibered/cofibered generators etc. We call an object A of a category $\mathcal{C}$ possess a fibered/cofibered structure or property over an object S of $\mathcal{C}$ if A has the same structure or property in $\mathcal{C}_S$/ $\mathcal{C}^S$. We study the properties of fibered and cofibered structures relative to original structures. We have , also investigated some general properties with respect to fibered/cofibered structures e.g. Theorem 3.3 , 3.4 , 3.5 , 3.6 , 3.8 , 3.16 , 3.17 , etc. and the proposition like Proposition 3.39 etc. might be of some interest. Notions like fibered essential morphisms , fibered injective hulls , and projective covers , fibered reflective subcategory etc. with their duals were introduced and studied.

3.1. Fibered and cofibered Projectives

In this section , we define fibered and cofibered projective objects over an object in a category,and observe that projective objects are fibered/cofibered projective objects over terminal/initial objects , and if an object is fibered/cofibered projective over some terminal/initial object then it should be projective. We also find that in V-categories all fibered and cofibered projective objects are projective. In the end, we characterize direct product and sum of fibered projectives.

Definition 3.1. Let $\mathcal{C}$ be a category and S be an object of $\mathcal{C}$. We say that an object A of $\mathcal{C}$ is <u>fibered projective</u> over an object S of $\mathcal{C}$ if it is projective in $\mathcal{C}_S$. i.e.

(i) there exists a morphism $\alpha : A \longrightarrow S$ in $\mathcal{C}$,

and

(ii) for every epimorphism $(A_1,\alpha_1) \xrightarrow{e} (A_2,\alpha_2)$ and every morphism $f : (A,\alpha) \longrightarrow (A_2,\alpha_2)$ in $\mathcal{C}_S$, there exists a unique morphism $\gamma : (A,\alpha) \longrightarrow (A_1,\alpha_1)$ in $\mathcal{C}_S$ such that the following diagram

$$
\begin{tikzcd}
 & (A,\alpha) \arrow[dl, dashed, "\gamma"'] \arrow[d, "f"] \\
(A_1,\alpha_1) \arrow[r, "e"] & (A_2,\alpha_2)
\end{tikzcd}
\quad \equiv \quad
\begin{tikzcd}
 & A \arrow[dl, dashed, "\gamma"'] \arrow[d, "f"] \arrow[dr, "\alpha"] & \\
A_1 \arrow[r, "e"] \arrow[rr, bend right, "\alpha_1"'] & A_2 \arrow[r, "\alpha_2"] & S
\end{tikzcd}
$$

is commutative.

Definition 3.2. An object A of a category $\mathcal{C}$ is called <u>cofibered projective</u> over an object S of $\mathcal{C}$ if it is projective in $\mathcal{C}^S$.

Proposition 3.1. Let $\mathcal{C}$ be a category. Then a projective object is fibered projective over terminal objects of $\mathcal{C}$.

Proof. Let P be a projective object of $\mathcal{C}$ and S be a terminal object. Then , since S is a terminal object , there exists a unique morphism $\alpha : P \longrightarrow S$. Next , if we consider the diagram

$$\begin{tikzcd} & (P,\alpha) \arrow[dl, dashed, "\Upsilon"'] \arrow[d, "f"] \\ (A_1,\alpha_1) \arrow[r, "e"] & (A_2,\alpha_2) \end{tikzcd} \qquad \ldots(3.a)$$

with epimorphism e in $\mathcal{C}_S$, and hence , by lemma 1.19 , $e : A_1 \longrightarrow A_2$ is epi in $\mathcal{C}$,implies that there exists a unique morphism $\Upsilon : P \longrightarrow A_1$ such that

$$P \xrightarrow{\Upsilon} A_1 \xrightarrow{e} A_2 = P \xrightarrow{f} A_2 .$$

Now , $\alpha_1\Upsilon = \alpha_2 e\Upsilon = \alpha_2 f = \alpha$

$\Longrightarrow \gamma : (P,\alpha) \longrightarrow (A_1,\alpha_1)$ belongs to $\mathcal{C}_S$ and diagram (3.a) is commutative. ||

Proposition 3.2. Let $\mathcal{C}$ be a V-category such that every epimorphism in $\mathcal{C}_S$ is an epimorphism in $\mathcal{C}$. Then a projective object is fibered projective over all objects of $\mathcal{C}$.

Proof. Let P be a projective object and S be any object of $\mathcal{C}$. Since $\mathcal{C}$ is a V-category, there exists a morphism $\alpha : P \longrightarrow S$ in $\mathcal{C}$. Also, since by hypothesis, every epimorphism in $\mathcal{C}_S$ is epi in $\mathcal{C}$ for every S, the result follows proceeding as in proof of proposition 3.1. ||

Proposition 3.3. If an object P is fibered projective over a terminal object, then it is projective.

Proof. Let P be a fibered projective object over a terminal object S. Consider the following diagram with epimorphism $e : A_1 \longrightarrow A_2$ in $\mathcal{C}$.

$$\begin{tikzcd} & P \arrow[d, "f"] \\ A_1 \arrow[r, "e"] & A_2 \end{tikzcd} \qquad \ldots(3.b)$$

Since S is a terminal object, we have the following diagram in $\mathcal{C}_S$

$$\begin{tikzcd} & (P,\alpha) \arrow[d, "f"] \\ (A_1,\alpha_1) \arrow[r, "e"] & (A_2,\alpha_2) \end{tikzcd}$$

Next , since e is epi in $\mathcal{C}$, hence , by lemma 1.19 , it is epi in $\mathcal{C}_S$. Therefore , there exists a unique morphism $\gamma : (P,\alpha) \longrightarrow (A_1,\alpha_1)$ such that

$$(P,\alpha) \xrightarrow{\gamma} (A_1,\alpha_1) \xrightarrow{e} (A_2,\alpha_2) = (P,\alpha) \xrightarrow{f} (A_2,\alpha_2)$$

Hence , there exists a unique morphism $\gamma : P \longrightarrow A_1$ in $\mathcal{C}$ such that

$$P \xrightarrow{\gamma} A_1 \xrightarrow{e} A_2 = P \xrightarrow{f} A_2$$

$\Longrightarrow$ P is projective in $\mathcal{C}$. ||

From proposition 3.1 and 3.3 the following theorem immediately follows :

Theorem 3.1. If S is a terminal object in $\mathcal{C}$, then $P \in \mathcal{C}$ is projective if and only if it is fibered projective over S .

Proposition 3.4. Let $\mathcal{C}$ be a V-category . Then every fibered projective object over any object is projective in $\mathcal{C}$.

Proof. Let P be a fibered projective object over an object S. Consider the following diagram with an epimorphism $e: A_1 \longrightarrow A_2$ in $\mathcal{C}$.

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, "f"] \\ A_1 \arrow[r, "e"] & A_2 \end{tikzcd} \qquad (3.e)$$

Since $\mathcal{C}$ is a V-category , there exist a morphism $\alpha_2 : A_2 \longrightarrow S$ and considering $\alpha = \alpha_2 f : P \longrightarrow S$ and $\alpha_1 = \alpha_2 e : A_1 \longrightarrow S$, we have the following diagram in $\mathcal{C}_S$

$$\begin{tikzcd} & (P,\alpha) \arrow[dl, dashed, "\Upsilon"'] \arrow[d, "f"] \\ (A_1,\alpha_1) \arrow[r, "e"] & (A_2,\alpha_2) \end{tikzcd} \qquad (3.d)$$

Since e is epi in $\mathcal{C}$, it is epi in $\mathcal{C}_S$. Hence there exists a unique morphism $\Upsilon : (P,\alpha) \longrightarrow (A_1,\alpha_1)$ such that the diagram (3.d) is commutative and hence, there exists a unique morphism $\Upsilon : P \longrightarrow A_1$ such that the diagram (3.e) is commutative $\Longrightarrow$ P is projective. ||

Remark 3.1. Because of the remark on Lemma 1.19 , every epimorphism in $\mathcal{C}_S$ is not necessarily an epimorphism in $\mathcal{C}$, we cannot find a true converse of proposition 3.4 like theorem 3.1.

Proposition 3.5. Every projective object is cofibered projective over an initial object.

Proof. Let P be a projective object and S be an initial object. Then there exists a unique morphism $p : S \longrightarrow P$.

Next , consider the following diagram

$$\begin{array}{ccc} & & (p,P) \\ & & \downarrow f \\ (\alpha_1,A_1) & \xrightarrow{\quad e \quad} & (\alpha_2,A_2) \end{array} \qquad (3.e)$$

with an epimorphism e in $\mathcal{C}^S$. Then $e : A_1 \longrightarrow A_2$ is an epimorphism in $\mathcal{C}$ by lemma 1.18 and hence , there exists a unique morphism $\gamma : P \longrightarrow A_1$ such that

$$P \xrightarrow{\gamma} A_1 \xrightarrow{e} A_2 = P \xrightarrow{f} A_2 .$$

Next , since S is initial object , $\gamma p = \alpha_1 \Longrightarrow \gamma : (p,P) \longrightarrow (\alpha_1,A_1)$ belongs to $\mathcal{C}^S$ such that the diagram (d.e) is commutative

$\Longrightarrow$ P is cofibered projective over S. ||

Proposition 3.6. A cofibered projective object over an initial object is projective.

Proof. Let P be a cofibered projective object over the initial object S. Then consider the following diagram

$$\begin{tikzcd} & P \arrow[d, "f"] \\ A_1 \arrow[r, "e"] & A_2 \end{tikzcd} \tag{3.f}$$

with $e : A_1 \longrightarrow A_2$ an epimorphism in $\mathcal{C}$. Since S is an initial object the diagram (3.f) induces the following diagram in $\mathcal{C}^S$

$$\begin{tikzcd} & (p,P) \arrow[dl, dashed, "\Upsilon"'] \arrow[d, "f"] \\ (\alpha_1,A_1) \arrow[r, "e"] & (\alpha_2,A_2) \end{tikzcd} \tag{3.g}$$

Since, by lemma 1.18, e is epi in $\mathcal{C}$, hence in $\mathcal{C}^S$. Therefore there exists a unique morphism $\Upsilon : (p,P) \longrightarrow (\alpha_1,A_1)$ in $\mathcal{C}^S$

such that the diagram (3.g) is commutative and hence there exists a unique $\Upsilon : P \longrightarrow A_1$ in $\mathcal{C}$ such that (3.f) is commutative.

From these two propositions 3.5 and 3.6 we have the following theorem :

Theorem 3.2. If S is an initial object, then P is projective if and only if P is cofibered projective over S.

The following is an interesting theorem which connects any fibered projective over an object with its direct factor.

Theorem 3.3. Let $\mathcal{C}$ be a category with products, and $S = \prod_{i \in I} S_i$ be a product of $\{S_i\}_{i \in I}$ in $\mathcal{C}$. Then an object P is fibered projective over S if and only if it is fibered projective over each S_i .

Proof. Let P be a fibered projective object over each S_i , this implies that there exists a family $\{p_i : P \longrightarrow S_i\}_{i \in I}$ of morphisms. Since S is product of the family $\{S_i\}_{i \in I}$, there exists a unique morphism $p : P \longrightarrow S$ such that

$P \xrightarrow{p} S \xrightarrow{s_i} S_i = P \xrightarrow{p_i} S_i$, where $s_i : S \longrightarrow S_i$, $\forall\, i \in I$, are canonical projection morphisms. Hence , we have a morphism $p : P \longrightarrow S$ in $\mathcal{C}$.

Next , if we consider the following diagram with an epimorphism e in $\mathcal{C}_S$

$$
\begin{tikzcd}
 & (P,p) \arrow[d, "f"] \\
(A_1,\alpha_1) \arrow[r, "e"] & (A_2,\alpha_2)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & P \arrow[d, "f"] \arrow[dr, "p"] & \\
A_1 \arrow[r, "e"] \arrow[rr, bend right, "\alpha_1"'] & A_2 \arrow[r, "\alpha_2"] & S
\end{tikzcd}
\qquad (3.h)
$$

then we have the following commutative diagrams in $\mathcal{C}_{S_i}, \forall i$.

$$
\begin{tikzcd}
 & P \arrow[d, "f"] \arrow[dr, "p"] \arrow[ddr, bend left, "s_i p"] & \\
A_1 \arrow[r, "e"] \arrow[rr, "\alpha_1"'] \arrow[rrd, "s_i\alpha_1"'] & A_2 \arrow[r, "\alpha_2"] & S \arrow[d, "s_i"] \\
 & & S_i
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & (P,s_i p) \arrow[dl, dashed, "\gamma"'] \arrow[d, "f"] \\
(A_1,s_i\alpha_1) \arrow[r] & (A_2,s_i\alpha_2)
\end{tikzcd}
\qquad (3.i)
$$

Next , since P is fibered projective over each S_i, there exists a unique morphism $\gamma : (P,s_i p) \longrightarrow (A_1,s_i\alpha_1)$ such that the diagram (3.i) is commutative in $\mathcal{C}_{S_i}$.

Moreover , $s_i p = s_i \alpha_1 \gamma$, $\forall i \in I$. By definition of the product , p is unique , hence $p = \alpha_1 \gamma \Longrightarrow \gamma : (P,p) \longrightarrow (A_1,\alpha_1) \in \mathcal{C}_S$ such that the diagram (3.h) is commutative $\Longrightarrow$ P is fibered projective over S.

Conversely , let P be fibered projective over S. Then we have a morphism $p : P \longrightarrow S$. Since $S = \pi S_i$, we have morphisms $p_i : P \longrightarrow S_i = P \xrightarrow{p} S \xrightarrow{s_i} S_i$, where

$s_i : S \longrightarrow S_i$, $\forall$ i , are canonical projections.

Next , if we consider the following diagram with an epimorphism $e : (A_1,\alpha_1) \longrightarrow (A_2,\alpha_2)$ in $\mathcal{C}_{S_i}$, $\forall i \in I$,

$$\begin{tikzcd} & (P,p_i) \arrow[d, "f"] \\ (A_1,\alpha_1) \arrow[r, "e"] & (A_2,\alpha_2) \end{tikzcd} \quad = \quad \begin{tikzcd} & P \arrow[d, "f"] \arrow[ddr, "p_i"] & \\ A_1 \arrow[r, "e"] \arrow[drr, "\alpha_{1,i}"'] & A_2 \arrow[dr, "\alpha_{2,i}"] & \\ & & S_i \end{tikzcd} \qquad (3.j)$$

and since $S = \prod S_i$, we have unique morphisms

$P \xrightarrow{p} S$ such that $P \xrightarrow{p} S \xrightarrow{u_i} S_i = P \xrightarrow{p_i} S_i$, $\forall i \in I$

$A_2 \xrightarrow{\alpha_2} S$ such that $A_2 \xrightarrow{\alpha_2} S \xrightarrow{u_i} S_i = A_2 \xrightarrow{\alpha_{2,i}} S_i$, $\forall i \in I$,

$A_1 \xrightarrow{\alpha_1} S$ such that $A_1 \xrightarrow{\alpha_1} S \xrightarrow{u_i} S_i = A_1 \xrightarrow{\alpha_{1,i}} S_i$, $\forall i \in I$, and hence we obtain the following diagram in $\mathcal{C}_S$.

$$\begin{tikzcd} & (P,p) \arrow[dl, dashed, "\gamma"'] \arrow[d] \\ (A_1,\alpha_1) \arrow[r] & (A_2,\alpha_2) \end{tikzcd} \tag{3.k}$$

Since P is fibered projective over S , there exists a unique morphism $\gamma : (P,p) \longrightarrow (A_1,\alpha_1)$ such that the diagram (3.k) is commutative.

Moreover , $P \xrightarrow{\gamma} A_1 \xrightarrow{\alpha_{1,i}} S_i = P \xrightarrow{\gamma} A_1 \xrightarrow{\alpha} S \xrightarrow{s_i} S_i$

$$= P \xrightarrow{p} S \xrightarrow{s_i} S_i = P \xrightarrow{p_i} S_i$$

$\Longrightarrow \gamma : (P,p_i) \longrightarrow (A_1,\alpha_{1,i})$ is unique morphism belongs to $\mathcal{C}_{S_i}$ such that the diagram (3.j) is commutative $\Longrightarrow$ P is fibered projective over each S_i . ||

The following Theorem can be proved dually.

Theorem 3.4. Let $\mathcal{C}$ be a category with coproducts and $S = \bigoplus_{i \in I} S_i$ be a coproduct of the family $\{S_i\}_{i \in I}$ of objects of $\mathcal{C}$. Then an object P is cofibered projective over S if and only if it is cofibered projective over each S_i.

The following theorem gives another characterization of fibered projectives over an object in $\mathcal{C}$.

Theorem 3.5. Let $\mathcal{C}$ be a V-category with coproducts and $P = \bigoplus_{i \in I} P_i$ be a coproduct of a family $\{P_i\}_{i \in I}$ of objects of $\mathcal{C}$. Then P is fibered projective over S if and only if each P_i, $\forall i \in I$, is fibered projective over S.

Proof. Let P be fibered projective over S. Then , there exists a morphism $p : P \longrightarrow S$. Then , we have morphisms $p_i : P_i \longrightarrow S = P_i \xrightarrow{u_i} P \xrightarrow{p} S$ in $\mathcal{C}$, $\forall i$, where u_i's are canonical injections.

Next , consider the following diagram with an epimorphism e in $\mathcal{C}_S$.

$$\begin{tikzcd} & (P_i, p_i) \arrow[d, "f_i"] \\ (A_1, \alpha_1) \arrow[r, "e"] & (A_2, \alpha_2) \end{tikzcd} \equiv \begin{tikzcd} & P_i \arrow[d, "f_i"'] \arrow[dr, "p_i"] & \\ A_1 \arrow[r, "e"] \arrow[rr, bend right, "\alpha_1"'] & A_2 \arrow[r, "\alpha_2"] & S \end{tikzcd} \qquad (3.c)$$

i .

Now, as $P = \bigoplus_{i \in I} P_i$, there exists unique morphisms

$f : P \longrightarrow A_2$, such that $P_i \xrightarrow{u_i} P \xrightarrow{f} A_2 = P_i \xrightarrow{f_i} A_2$, $\forall i \in I$,
and

$p : P \longrightarrow S$, such that $P_i \xrightarrow{u_i} P \xrightarrow{p} S = P_i \xrightarrow{p_i} S$, $\forall i \in I$,
hence, we have the following commutative diagram in $\mathscr{C}$

$$\begin{tikzcd}
 & P_i \arrow[dl, dashed, "\gamma"'] \arrow[d, "u_i"'] \arrow[dd, bend left, "f_i"] \arrow[ddrr, "p_i"] & & \\
 & P \arrow[d, "f"] \arrow[drr, "p"] & & \\
A_1 \arrow[r, "e"] \arrow[rrr, bend right, "\alpha_1"'] & A_2 \arrow[rr, "\alpha_2"] & & S
\end{tikzcd} \qquad (3.m)$$

$\Longrightarrow \alpha_2 f u_i = \alpha_2 f_i \alpha p_i = p u_i \Longrightarrow \alpha_2 f = p$, hence we have the following diagram

$$\begin{tikzcd}
 & (P,p) \arrow[dl, dashed, "\gamma"'] \arrow[d, "f"] \\
(A_1,\alpha_1) \arrow[r, "e"] & (A_2,\alpha_2)
\end{tikzcd} \qquad (3.n)$$

in $\mathscr{C}_S$. Now, since P is fibered projective over S, there exists a unique morphism $\gamma : (P,p) \longrightarrow (A_1,\alpha_1)$ in $\mathcal{C}_S$ such that the diagram (3.n) is commutative.

Now, define $\gamma_i : P_i \longrightarrow A_1 = P_i \xrightarrow{u_i} P \xrightarrow{\gamma} A$, $\forall i$, which are unique, for γ is unique.

Also, $\alpha_1\gamma_i = \alpha_1\gamma u_i = pu_i = p_i$.

Therefore, $\gamma_i : (P_i,p_i) \longrightarrow (A_1,\alpha_1) \in \mathcal{C}_S$, which makes the diagram (3. ( ) commutative.

Conversely, let as suppose that each $\{P_i\}_{i \in I}$, is fibered projective over S $\Longrightarrow$ there exists a family $\{P_i \xrightarrow{p_i} S\}_{i \in I}$ of morphisms. Since P is coproduct of $\{P_i\}_{i \in I}$, there exists a unique morphism $p : P \longrightarrow S$ such that

$$P_i \xrightarrow{u_i} P \xrightarrow{p} S = P_i \xrightarrow{p_i} S, \ \forall i \in I.$$

This satisfies the first condition of the definition.

Next, if we consider the following diagram with epimorphism e in $\mathcal{C}_S$

$$\begin{tikzcd} & (P,p) \arrow[d, "f"] \\ (A_1,\alpha_1) \arrow[r, "e"] & (A_2,\alpha_2) \end{tikzcd} \quad = \quad \begin{tikzcd} & P \arrow[d, "f"] \arrow[dr, "p"] & \\ A_1 \arrow[r, "e"] \arrow[rr, bend right, "\alpha_1"'] & A_2 \arrow[r, "\alpha_2"] & S \end{tikzcd} \qquad (3.o)$$

Then we have the following diagrams , $\forall i \in I$, in $\mathcal{C}_S$

$$\begin{tikzcd}
 & P_i \arrow[d, "u_i"'] \arrow[ddr, "pu_i"] & \\
 & P \arrow[d, "f"'] \arrow[dr, "p"] & \\
A_1 \arrow[r, "e"] \arrow[rr, bend right, "\alpha_1"'] & A_2 \arrow[r, "\alpha_2"] & S
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & (P_i, pu_i) \arrow[d, "fu_i"] \\
(A_1, \alpha_1) \arrow[r, "e"] & (A_2, \alpha_2)
\end{tikzcd} \qquad (3.p)$$

Now , since P_i's are fibered projectives over S , there exists morphisms $\gamma_i : (P_i, pu_i) \longrightarrow (A_1, \alpha_1)$ in $\mathcal{C}_S$, $\forall i$, such that the diagram (3.p) is commutative. Also , since $P = \oplus \; P_i$, there exists a unique morphism $\gamma : P \longrightarrow A_1$ such that

$$P \xrightarrow{\gamma} A_1 \xrightarrow{e} A_2 = P \xrightarrow{f} A_2$$

Now, $P \xrightarrow{\gamma} A_1 \xrightarrow{\alpha_1} S = P \xrightarrow{\gamma} A_1 \xrightarrow{e} A_2 \xrightarrow{\alpha_2} S = P \xrightarrow{f} A_2 \xrightarrow{\alpha_2} S = P \xrightarrow{p} S$

$\Longrightarrow \gamma : (P, p) \longrightarrow (A_1, \alpha_1)$ is a unique morphism in $\mathcal{C}_S$ such that the diagram (3.e) is commutative.

3.2. Fibered and cofibered injectives

In this section , fibered and cofibered injective objects over an object of a category are defined dually to what is done mostly in the section 3.2. We find that injective objects are fibered / cofibered injectives over terminal/initial objects, also if an object is fibered/cpfibered injectives over some terminal/initial object, then it is injective. We further obtain that in V-categories , all fibered and cofibered injectives are injectives. We close the section with the characterizations of fibered injectives.

Definition 3.3. Let $\mathcal{C}$ be a category and S be an object of $\mathcal{C}$, we say that an object Q of $\mathcal{C}$ is <u>fibered injective</u> over S if it is injective in $\mathcal{C}_S$,i.e.

(i) there exists a morphism $q : Q \longrightarrow S$,

and

(ii) for every monomorphism $(A_1,\alpha_1) \xrightarrow{m} (A_2,\alpha_2)$ and every morphism $f : (A_1,\alpha_1) \longrightarrow (Q,q)$ in $\mathcal{C}_S$, there exists a unique morphism $\gamma : (A_2,\alpha_2) \longrightarrow (Q,q)$ in $\mathcal{C}_S$ such that the following diagram

$$\begin{tikzcd}
(A_1,\alpha_1) \arrow[rr, "m"] \arrow[d, "f"] & & (A_2,\alpha_2) \arrow[dll, dashed, "\gamma"] \\
(Q,q) & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
S & & \\
A_1 \arrow[u, "\alpha_1"] \arrow[rr, "m"] \arrow[d, "f"] & & A_2 \arrow[ull, "\alpha_2"'] \arrow[dll, dashed, "\gamma"] \\
Q \arrow[uu, bend left] & &
\end{tikzcd}$$

is commutative.

Definition 3.4. An object Q of a category $\mathcal{C}$ is called <u>cofibered injective</u> over an object S if it is injective in $\mathcal{C}^S$.

The following are the relations between injectives and cofibered injective.

Proposition 3.7. An injective object is fibered injective over terminal object.

Proof. Let Q be an injective object and S be a terminal object of $\mathcal{C}$. Since S is terminal object , there exists a unique morphism $q : Q \longrightarrow S$ in $\mathcal{C}$. Next , consider the following diagram with a monomorphism m in $\mathcal{C}_S$.

$$\begin{tikzcd} (A_1,\alpha_1) \arrow[r, "m"] \arrow[d, "f"'] & (A_2,\alpha_2) \arrow[dl, dashed, "\gamma"] \\ (Q,q) & \end{tikzcd} \equiv \begin{tikzcd} & S & \\ & A_1 \arrow[u, "\alpha_1"'] \arrow[r, "m"] \arrow[d, "f"] & A_2 \arrow[ul, "\alpha_2"'] \\ & Q \arrow[uu, bend left, "q"] & \end{tikzcd} \qquad (3.q)$$

Since $m \in \mathcal{C}_S$ is mono in $\mathcal{C}_S$, by lemma 1.18, it is mono in $\mathcal{C}$ ==> there exists a unique morphism $\gamma : A_2 \longrightarrow Q$ such that

$$A_1 \xrightarrow{m} A_2 \xrightarrow{\gamma} Q = A_1 \xrightarrow{f} Q \text{ in } \mathcal{C} .$$

Also , since S is terminal object , $q\gamma = \alpha_2 \Longrightarrow \gamma : (A_2, \alpha_2) \longrightarrow (Q, q)$ belongs to $\mathcal{C}_S$, which makes diagram (3.q) commutative ==> Q is fibered injective over S. ||

Proposition 3.8. If Q is fibered injective over some terminal object S , then it is injective.

Proof. Let Q be fibered injective over a terminal object S ,and let us have the following diagram with a monomorphism m in $\mathcal{C}$.

$$\begin{tikzcd} A_1 \arrow[r, "m"] \arrow[d, "f"] & A_2 \\ Q & \end{tikzcd} \qquad (3.r)$$

Since S is a terminal object , the diagram (3.r) induces the following diagram in $\mathcal{C}_S$.

$$\begin{tikzcd} (A_1, \alpha_1) \arrow[r, "m"] \arrow[d, "f"] & (A_2, \alpha_2) \\ (Q, q) & \end{tikzcd} \qquad (3.s)$$

Now , since Q is fibered injective over S , there exists a unique morphism $\gamma : (A_2, \alpha_2) \longrightarrow (Q, q)$ in $\mathcal{C}_S$ such that diagram(3.s) is commutative , and hence , we have a unique $\gamma : A_2 \longrightarrow Q$ in $\mathcal{C}$ such that the diagram (3.r) is commutative $\Longrightarrow$ Q is injective.||

The above two propositions 3.7 and 3.8 imply :

Theorem 3.6. An object is injective if and only if it is fibered injective over a terminal object.

The following proposition deel with the relations between injectives and cofibered injectives.

Proposition 3.9. An injective object is cofibered injective over an initial object.

Proof. Let Q be an injective object and S be an initial object in $\mathcal{C}$. Since S is initial object , there exists a morphism $q : S \longrightarrow Q$ in $\mathcal{C}$. Next consider the following diagram with a monomorphism m in $\mathcal{C}^S$.

$$\begin{tikzcd}
(\alpha_1, A_1) \arrow[r, "m"] \arrow[d, "f"'] & (\alpha_2, A_2) \arrow[dl, dashed, "\gamma"] \\
(q, Q) &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
S \arrow[dr, "\alpha_1"] \arrow[drr, "\alpha_2"] \arrow[ddr, "q"'] & & \\
& A_1 \arrow[r, "m"] \arrow[d, "f"] & A_2 \\
& Q &
\end{tikzcd}
\qquad (3.t)$$

Since m is mono in $\mathcal{C}^S$, by lemma 1.19 , it is mono in $\mathcal{C}$ and since Q is injective , there exists a unique morphism $\Upsilon : A_2 \longrightarrow Q$ in $\mathcal{C}$ such that

$$A_1 \xrightarrow{m} A_2 \xrightarrow{\Upsilon} Q = A_1 \xrightarrow{f} Q .$$

Moreover , $\alpha\Upsilon_2 = \Upsilon m\alpha_1 = f\alpha_1 = q$.

$\Longrightarrow \Upsilon : (\alpha_2, A_2) \longrightarrow (q, Q)$ is a unique morphism in $\mathcal{C}^S$ such that the diagram (3.t) is commutative $\Longrightarrow$ Q is cofibered injective over S. ||

Proposition 3.10. A cofibered injective object over an initial object is injective.

Proof. Let Q be cofibered injective object over an initial object S of $\mathcal{C}$, and consider the following diagram with a monomorphism m in $\mathcal{C}$.

$$\begin{tikzcd} A_1 \arrow[r, "m"] \arrow[d, "f"] & A_2 \arrow[dl, dotted, "\Upsilon"] \\ Q & \end{tikzcd} \qquad (3.u)$$

Since S is initial object , the diagram (3.u) induces the following diagram in $\mathcal{C}^S$

$$\begin{tikzcd}
(\alpha_1, A_1) \arrow[r, "m"] \arrow[d, "f"] & (\alpha_2, A_2) \arrow[dl, dotted, "\gamma"] \\
(q, Q) &
\end{tikzcd} \tag{3.v}$$

Now, since m is mono in $\mathcal{C}$, by lemma 1.19, it is mono in $\mathcal{C}^S$ and since Q is cofibered injective over S, there exists a unique morphism $\gamma : (\alpha_2, A_2) \longrightarrow (q, Q)$ which leaves the diagram (3.v) commutative, and hence, there exists a unique morphism $\gamma : A_2 \longrightarrow Q$ in $\mathcal{C}$, which leaves the diagram (3.u) commutative. ||

These two propositions 3.9 and 3.10 imply :

Theorem 3.7. An object is injective if and only if it is cofibered injective over an initial object of the category.

In V-category, we have following relation between injectivity and cofibered injectivity.

Proposition 3.11. Let $\mathcal{C}$ be a V-category such that every monomorphism in $\mathcal{C}^S$ for every S is monomorphism in $\mathcal{C}$. Then an injective object is cofibered injective over all objects of $\mathcal{C}$.

Proof. Let Q be an injective object in $\mathcal{C}$ and S be an object of $\mathcal{C}$. Since $\mathcal{C}$ is a V-category, $[S, Q] \neq \emptyset$. Also since every monomorphism in $\mathcal{C}^S$ is a monomorphism in $\mathcal{C}$, therefore result follows proceeding as in proposition 3.9. ‖

Proposition 3.12. In V-categories, every cofibered injective object over an object S is injective.

Proof. Let Q be cofibered injective over an object S, and consider the diagram (3.u) with monomorphism m in $\mathcal{C}$. Since $\mathcal{C}$ is a V-category, there exists a morphism $\alpha_1 : S \longrightarrow A_1$ in $\mathcal{C}$. Then considering $\alpha_2 = m\alpha_1 : S \longrightarrow A_2$ and $q = f\alpha_1 : S \longrightarrow Q$, we have diagram (3.v) in $\mathcal{C}^S$. Since $m : A \longrightarrow B$ is mono in $\mathcal{C}$, by lemma 1.19, $m : (\alpha_1, A_1) \longrightarrow (\alpha_2, A_2)$ is also mono in $\mathcal{C}^S$. Therefore, there exists a unique morphism $\gamma : (\alpha_2, A_2) \longrightarrow (q, Q)$ such that the diagram (3.v) is commutative, and hence we have a unique $\gamma : A_2 \longrightarrow Q$ in $\mathcal{C}$ such that

$$A_1 \xrightarrow{m} A_2 \xrightarrow{\gamma} Q = A_1 \xrightarrow{f} Q \quad .$$

Hence Q is injective. |

The following theorems can be proved similarly as we have proved theorems 3.3 and 3.4 for the projective case :

Theorem 3.8. Let $\mathcal{C}$ be a category with products and $S = \prod_{i \in I} S_i$. Then an object Q is fibered injective over S if and only if it is fibered injective over each S_i.

Theorem 3.9. Let $\mathcal{C}$ be a category with coproducts and $S = \bigoplus_{i \in I} S_i$ be a coproducts of the family $\{S_i\}_{i \in I}$. Then an object Q is cofibered injective over S if and only if it is cofibered injective over each S_i .

The following theorem gives the characterization of fibered/cofibered injectives over an object , which is dual to the case of fibered/cofibered projectives over an object (Theorem 3.5).

Theorem 3.10. Let $\mathcal{C}$ be a category with products and $Q = \prod_{i \in I} Q_i$. Then Q is fibered injectives over S if and only if each Q_i is fibered injective over S.

3.3. Fibered and cofibered essential epimorphisms and monomorphisms

In this section , we define fibered and cofibered essential epimorphisms and monomorphisms and observe that , in general , in V-categories, essential epimorphisms/ monomorphisms are cofibered / fibered essential epimorphism/monomorphism ; however , in certain restricted V-categories , essential epimorphisms / monomorphisms are both fibered and cofibered essential epimorphisms/monomorphisms over an object. Finally, we show that over a terminal object , every essential epimorphisms/monomorphisms is fibered essential epimorphism/ monomorphism and conversely ; over an initial object every essential epimorphism / monomorphism is cofibered essential epimorphism / monomorphism.

Definition 3.5. An epimorphism $e : A \longrightarrow B$ in $\mathcal{C}$ is called <u>fibered / cofibered essential epimorphism</u> over an object S if e is an essential epimorphism in $\mathcal{C}_S$ / $\mathcal{C}^S$ i.e. e is fibered essential epimorphism if (i) there exist morphisms $\alpha : A \longrightarrow S$ and $\beta : B \longrightarrow S$ such that $A \xrightarrow{e} B \xrightarrow{\beta} S = A \xrightarrow{\alpha} S$, and (ii) for some morphism g in $\mathcal{C}_S$, if eg is an epimorphism in $\mathcal{C}_S$, then g is an epimorphism in $\mathcal{C}_S$. Similarly for the other case.

Definition 3.6. A monomorphism $m : A \longrightarrow B$ in $\mathcal{C}$ is called <u>fibered/cofibered essential monomorphism</u> over an object S if $m : A \longrightarrow B$ is essential monomorphism in $\mathcal{C}_S$/ $\mathcal{C}^S$. That is, in first case , if (i) there exist morphisms $\alpha : A \longrightarrow S$ and

$\beta : B \longrightarrow S$ such that $A \xrightarrow{m} B \xrightarrow{\beta} S = A \xrightarrow{\alpha} S$ and (ii) for some morphism n in $\mathcal{C}_S$, if nm is monomorphism in $\mathcal{C}_S$, then n is a monomorphism in $\mathcal{C}_S$, similarly for the other case.

Proposition 3.13. Let $\mathcal{C}$ be a V-category such that every epimorphism in $\mathcal{C}_S$ is an epimorphism in $\mathcal{C}$. Then an essential epimorphism is fibered essential epimorphism over the object S.

Proof. Let $A \xrightarrow{e} B$ be an essential epimorphism in $\mathcal{C}$. Since $\mathcal{C}$ is a V-category, there exists a morphism $\beta : B \longrightarrow S$ and, then defining $\alpha = \beta e : A \longrightarrow S$, we have $A \xrightarrow{e} B \xrightarrow{\beta} S = A \xrightarrow{\alpha} S$. Thus, condition (i) of definition 3.5 is satisfied. Now, if eg is an epimorphism in $\mathcal{C}_S$, then eg is epi in $\mathcal{C}$ by hypothesis and hence g is epi in $\mathcal{C}$ because e is an essential epimorphisms. Thus, by lemma 1.19, g is epi in $\mathcal{C}_S$. ||

Proposition 3.14. In a V-category, every essential epimorphism is cofibered essential epimorphism over any object of the category.

Proof. Let $e : A \longrightarrow B$ be an essential epimorphism. As $\mathcal{C}$ is a V-category, there exists a morphism $\alpha : S \longrightarrow A$ and then, if we consider $\beta = e\alpha : S \longrightarrow B$, the condition (i) of definition 3.5 is satisfied. Next, if e.g is an epimorphism in $\mathcal{C}^S$ for some morphism g in $\mathcal{C}^S$. Then, by lemma 1.18, it is an epimorphis in $\mathcal{C}$ and hence by definition of essential epimorphism g is an epimorphism in $\mathcal{C}$. Therefore , by lemma 1.19 g is epi in $\mathcal{C}^S$.

Proposition 3.15. Let $\mathcal{C}$ be a V-category such that every epimorphism in $\mathcal{C}_S$ is an epimorphism in $\mathcal{C}$. Then a fibered essential epimorphism over S is an essential epimorphism.

Proof. Let $A \xrightarrow{e} B$ is fibered essential epimorphism over S and $C \xrightarrow{g} A \xrightarrow{e} B$ is an epimorphism in $\mathcal{C}$. Since $A \xrightarrow{e} B$ is fibered essential epimorphism , there exist morphisms $\alpha : A \longrightarrow S$ and $\beta : B \longrightarrow S$ such that $A \xrightarrow{e} B \xrightarrow{\beta} S = A \xrightarrow{\alpha} S$.
If we define $\gamma = \alpha g : C \longrightarrow S$, then $g : (C,\gamma) \longrightarrow (A,\alpha) \in \mathcal{C}_S$ and, by lemma 1.19, eg is an epimorphism in $\mathcal{C}_S$. Therefore , g is an epimorphism in $\mathcal{C}_S$ and hence , by hypothesis , g is an epimorphism in $\mathcal{C}$. Therefore e is an essential epimorphism. ||
We have a dual Proposition to 3.14 , which is obviously as follows :

Proposition 3.16. Let $\mathcal{C}$ be a V-category. Then an essential monomorphism is a fibered essential monomorphism over any object S in $\mathcal{C}$.

Proposition 3.17. Let $\mathcal{C}$ be a V-category such that every monomorphism in $\mathcal{C}^S$ is a monomorphism in $\mathcal{C}$. Then an essential monomorphism is cofibered essential monomorphism over S.

Proof. Let $m : A \longrightarrow B$ be an essential monomorphism in $\mathcal{C}$. Since $\mathcal{C}$ is a V-category, there exists $\alpha : S \longrightarrow A$ and $\beta : S \longrightarrow B$ such that $S \longrightarrow A \longrightarrow B = S \longrightarrow B$. Next, if $n\,m$ is monomorphism in $\mathcal{C}^S$ for some morphism n in $\mathcal{C}^S$, then by hypothesis, it is mono in $\mathcal{C}$ implies n is mono in $\mathcal{C}$ and hence in $\mathcal{C}^S$, by lemma 1.19. Thus m is cofibered essential epimorphism. This is duallizes 3.15. ||

Proposition 3.18. Let $\mathcal{C}$ be a V-category such that every monomorphism in $\mathcal{C}^S$ is monomorphism in $\mathcal{C}$. Then a cofibered essential monomorphism over the object S is essential monomorphism.

Proof. Let $A \xrightarrow{m} B$ be a cofibered essential monomorphism and $A \xrightarrow{m} B \xrightarrow{n} C$ be a monomorphism in $\mathcal{C}$. Since m is cofibered essential monomorphism, there exist morphisms $\alpha : S \longrightarrow A$ and $\beta : S \longrightarrow B$ such that $S \xrightarrow{\alpha} A \xrightarrow{m} B = S \xrightarrow{\beta} B$. Then, if we define $\gamma : S \longrightarrow C = S \xrightarrow{\beta} B \xrightarrow{n} C$, then $nm : (\alpha, A) \longrightarrow (\gamma, C) \in \mathcal{C}^S$ and is a monomorphism. This implies n is a monomorphism in $\mathcal{C}^S$ and by hypothesis, n is monomorphism in $\mathcal{C}$.

$\Longrightarrow A \xrightarrow{m} B$ is essential monomorphism. ||

Propositions 3.13 , 3.15, 3.17 and 3.18 imply the following :

Theorem 3.11. Let $\mathcal{C}$ be a category such that every epimorphism/ monomorphism in $\mathcal{C}_S$ / $\mathcal{C}^S$ is epimorphism/monomorphism in $\mathcal{C}$. Then a morphism f is essential epimorphism / monomorphism if and only if f is fibered/cofibered essential epimorphism/ monomorphism over the object S.

If S is terminal or initial object then condition of propositions 3.13, 315, 3.19 and 3.18 are fulfilled for the object S and therefore , we have the following theorems :

Theorem 3.12. A morphism in $\mathcal{C}$ is essential epimorphism/ monomorphism if and only if it is fibered/cofibered essential epimorphism/monomorphism over every terminal/initial object S in $\mathcal{C}$.

Theorem 3.13. A morphism in $\mathcal{C}$ is essential monomorphism/ epimorphism if and only if it is fibered/cofibered essential monomorphism/epimorphism over every terminal/initial object S in $\mathcal{C}$.

3.4. Fibered and cofibered injective hulls and projective covers

In this section , we define fibered and cofibered projective covers and injevtive hulls over some object of a category and find that in certain restricted V-categories , if an object has projective cover/ injective hull, then it has fibered and cofibered projective cover/ injective hull. Further , if any two objects have projective covers over an object, then their product also has fibered projective cover over the same object.

Definition 3.7. An object A of a category $\mathcal{C}$ is said to have a fibered/cofibered projective oover over an object S if A has a projective oover in $\mathcal{C}_S / \mathcal{C}^S$. That is , in the first case, A has a fibered projective cover over an object S if (i) there exists a morphism $\alpha : A \longrightarrow S$, and (ii) there exists a fibered projective object P in $\mathcal{C}$ with a fibered essential epimorphism $P \longrightarrow A$.

Definition 3.8. An object A of a category $\mathcal{C}$ is said to have a fibered/cofibered injective hull if A has injective hull in $\mathcal{C}_S / \mathcal{C}^S$.

Proposition 3.19. Let $\mathcal{C}$ be a V-category such that every epimorphism in $\mathcal{C}_S$ is an epimorphism in $\mathcal{C}$. Then an object A has fibered projective cover over S if A has a projective cover in $\mathcal{C}$.

Proof. Let $P \xrightarrow{e} A$ be a projective cover of A in $\mathcal{C}$. Since $\mathcal{C}$ is a V-category, there exists a morphism $\alpha : A \longrightarrow S$, and then define $p = \alpha e : P \longrightarrow S$. Then we have $P \longrightarrow A \longrightarrow S = P \longrightarrow S$. Next, by proposition 3.2, P is fibered projective object over S and, by proposition 3.16, $P \xrightarrow{e} A$ is fibered essential epimorphism over S. Thus, $P \xrightarrow{e} A$ is fibered projective cover of A over S. ||

Proposition 3.20. Let $\mathcal{C}$ be a V-category such that every epimorphism in $\mathcal{C}_S$ is an epimorphism in $\mathcal{C}$. Then an object A has projective cover if A has fibered projective cover over S.

Proof. Let $P \xrightarrow{e} A$ be a fibered projective cover over S. Then, by proposition 3.4, P is projective for V-category $\mathcal{C}$ and, by proposition 3.15, e is an essential epimorphism. Thus, $P \xrightarrow{e} A$ is projective cover. ||

The above propositions 3.19 and 3.20 imply :

Theorem 3.14. In a V-category such that every epimorphism in $\mathcal{C}_S$ is epimorphism in $\mathcal{C}$, an object has projective cover if and only if it has fibered projective cover over S.

The following are the similar situations for injective hulls :

Proposition 3.21. Let $\mathcal{C}$ be a V-category such that every monomorphism in $\mathcal{C}^S$ is monomorphism in $\mathcal{C}$. Then an object A has cofibered injective hull over S if A has injective hull in $\mathcal{C}$.

Proof. Let $A \xrightarrow{m} Q$ be a an injective hull of A in $\mathcal{C}$. Then, by proposition 3.11 , Q is cofibered injective over S and , by proposition 3.17 , m is cofibered essential monomorphism over S. Thus , $A \xrightarrow{m} Q$ is cofibered injective hull of A over S . ||

Proposition 3.22. Let $\mathcal{C}$ be a V-category such that every monomorphism in $\mathcal{C}^S$ is a monomorphism in $\mathcal{C}$. Then an object A has an injective hull if A has cofibered injective hull over the object S.

Proof. Let $A \xrightarrow{m} Q$ be a cofibered injective hull of A over S. Then Q is injective by proposition 3.12 for all V-categories , and, by proposition 3.18, m is essential monomorphism. Hence : $m : A \longrightarrow Q$ is injective hull . ||

Both the above propositions 3.21 and 3.22 imply the following theorem :

Theorem 3.15. Let $\mathcal{C}$ be a V-category such that every monomorphism in $\mathcal{C}^S$ is a monomorphism in $\mathcal{C}$. Then an object A of $\mathcal{C}$ has injective hull if and only if A has cofibered injective hull over S.

The conditions of theorem 3.13/ 3.14 are also satisfied if we consider S to be a terminal/ an initial object respectively. Thus , we have the following theorem :

Theorem 3.16. An object A of a category $\mathcal{C}$ has projective cover/ injective hull if and only if A has fibered/cofibered projective cover/ injective hull over a terminal / an initial object of $\mathcal{C}$.

In order to prove that coproduct of fibered injective hulls over an object is a fibered injective hull over the object . We, first prove the following lemma :

Lemma 3.1. Let $\mathcal{C}$ be a category with corproducts . Then coproduct of two fibered essential monomorphisms over an object S is fibered essential monomorphism over S.

Proof. Let $A_1 \xrightarrow{m_1} B_1$ and $A_2 \xrightarrow{m_2} B_2$ be two fibered essential monomorphisms over an object S and $A_1 \oplus A_2 \xrightarrow{m_1 \oplus m_2} B_1 \oplus B_2$

be their coproduct . Then we have the following commutative diagrams with u_i and u'_i , $i = 1,2$, as canonical injections:

$$\begin{tikzcd}
A_1 \arrow[r, "m_1"] \arrow[d, "u_1"'] \arrow[drr, dashed, "\alpha_i"] & B_1 \arrow[d, "u_i"] \arrow[dr, "\beta_1"] & \\
A_1 \oplus A_2 \arrow[r, "m_1 \oplus m_2"] \arrow[rr, bend right, "\eta"'] & B_1 \oplus B_2 \arrow[r, "\xi"] & S
\end{tikzcd} \qquad (3.w)$$

Next , let the composition morphism

$$(A_1 \oplus A_2, \eta) \xrightarrow{m_1 \oplus m_2} (B_1 \oplus B_2, \xi) \xrightarrow{n} (C,\gamma)$$

be a monomorphism in $\mathcal{C}_S$. Then $n(m_1 \oplus m_2)\, u_i$, $i = 1,2$, are monomorphisms in $\mathcal{C}_S$. This implies, by commutativity of (3.w), $n\, u'_i m_i$, $i = 1,2$, are monomorphism in $\mathcal{C}_S$. Thus nu'_i , $i=1,2$, are monomorphisms in $\mathcal{C}_S$, since m_i, $i = 1,2$, are fibered essential monomorphisms , ==> nu'_i = $i=1,2$, are monomorphisms in $\mathcal{C}$, by lemma 1.18 . This implies that n is mono in $\mathcal{C}$, hence by lemma 1.18 , n is monomorphism in $\mathcal{C}_S$ ==> $m_1 \oplus m_2$ is fibered essential monomorphism. ||

Dually , we have the following lemma :

Lemma 3.2. Let $\mathcal{C}$ be a category with products. Then product of two cofibered essential epimorphisms over an object S is cofibered essential epimorphism over S.

Now the following theorems follows by propositions 3.23 and 3.24 and Lemmas 3.1 and 3.2 respectively.

Theorem 3.17. If objects A_1 and A_2 in a category $\mathcal{C}$ have fibered injective hulls Q_1 and Q_2 over an object S in $\mathcal{C}$ then $A_1 \oplus A_2$ has fibered injective hull over S.

Theorem 3.18. If A_1 and A_2 in a category $\mathcal{C}$ have cofibered projective covers over S , then $\prod A_i$, $i = 1,2$, has cofibered projective cover over S.

3.5. Fibered and cofibered generators and cogenerators

In this section , we define fibered and cofibered generators and cogenerators over an object in a category and find that a family of generators/cogenerators is a family of fibered generators/cofibered cogenerators and the converse is true if the category under consideration is a V-category with coequalizers/equalizers. Also , we observe that if $\mathcal{C}$ is a V-category with coproducts , then a family $\{U_i\}_{i \in I}$ is of fibered generators over S if and only if $\bigoplus_{i\in I} U_i$ is a fibered generator over S. In the end, we show that if $S = \prod S_i$ is a product, then an object U is a fibered generator over S if and only if U is a fibered generator over each S_i, $\forall i$.

Definition 3.9. In a category $\mathcal{C}$, we say that a family $\{U_i\}_{i\in I}$ is a family of <u>fibered/cofibered generators</u> over an object S of $\mathcal{C}$ if the family $\{U_i\}_{i\in I}$ is a family of generators in $\mathcal{C}_S / \mathcal{C}^S$. That is, in the first case, (i) there exists morphisms $u_i : U_i \longrightarrow S$, $\forall i$ and (ii) for every pair of different morphisms $\gamma_1, \gamma_2 : (A_1, \alpha_1) \longrightarrow (A_2, \alpha_2)$ in $\mathcal{C}_S$, there exists a morphism $u : (U_i, u_i) \longrightarrow (A_1, \alpha_1)$ for some i such that $\gamma_1 u \neq \gamma_2 u$ in $\mathcal{C}_S$.

Definition 3.10. In a category $\mathcal{C}$, we say that a family $\{U_i\}_{i\in I}$ is a family of <u>fibered/cofibered cogenerators</u> over an object S of $\mathcal{C}$ if the family $\{U_i\}_{i\in I}$ is a family of cogenerators in $\mathcal{C}_S / \mathcal{C}^S$.

If the family $\{U_i\}_{i\in I}$ has only one object U(say), then we say that U is fibered or cofibered generator or cogenerator over S.

Proposition 3.25. Let $\mathcal{C}$ be a V-category. If $\{U_i\}_{i\in I}$ is a family of generators of $\mathcal{C}$, then it is a family of fibered generators over all objects of $\mathcal{C}$.

Proof. Since $\mathcal{C}$ is a V-category, there exists morphisms $u_i : U_i \longrightarrow S$ for all S in $\mathcal{C}$. Next, let $\gamma_1 \neq \gamma_2 : (A_1, \alpha_1) \longrightarrow (A_2, \alpha_2)$ be two morphisms in $\mathcal{C}_S$, then $\gamma_1 \neq \gamma_2 : A_1 \longrightarrow A_2$ in $\mathcal{C} \Longrightarrow$ there exists a morphism $u : U_i \longrightarrow A_1$, for some i, such that $\gamma_1 u \neq \gamma_2 u$ in $\mathcal{C}$. Now, considering $u_i = \alpha_1 u : U_i \longrightarrow S$, we have $u : (U_i, u_i) \longrightarrow (A_1, \alpha_1)$ belings to $\mathcal{C}_S$ such that $\gamma_1 u \neq \gamma_2 u$ in $\mathcal{C}_S$. ||

Proposition 3.26. Let $\mathcal{C}$ be a V-category with coequalizers. Then a family $\{U_i\}_{i \in I}$ of fibered generators over an object S is a family of generators of $\mathcal{C}$.

Proof. Let $\gamma_1 \neq \gamma_2 : A_1 \longrightarrow A_2$ be two morphisms in $\mathcal{C}$. Let $A_2 \xrightarrow{\mu} C$ = coequalizers of γ_1 and γ_2. Since $\mathcal{C}$ is a V-category, there exists a morphism $\xi : C \longrightarrow S$. Then define $\alpha_2 = \xi\mu : A_2 \longrightarrow S$. Moreover, $\mu\gamma_1 = \mu\gamma_2 \Longrightarrow \xi\mu\gamma_1 = \xi\mu\gamma_2$. Therefore, define $\alpha_1 = \xi\mu\gamma_i : A_1 \longrightarrow S \Longrightarrow \gamma_i : (A_1, \alpha_1) \longrightarrow (A_2, \alpha_2)$ belongs to $\mathcal{C}_S$. Hence, since $\{U_i\}_{i \in I}$ a family of fibered generators , there exists a morphism $u:(U_i,u_i)\longrightarrow(A_1,\alpha_1)$ such that

$$(U_i,u_i)\xrightarrow{u}(A_1,\alpha_1)\xrightarrow{\Upsilon_1}(A_2,\alpha_2) \neq (U_i,u_i)\xrightarrow{u}(A_1,\alpha_1)\xrightarrow{\Upsilon_2}(A_2,\alpha_2)$$

and therefore , there exists $u : U_i \longrightarrow A_1$ in $\mathcal{C}$ such that

$$U_i\xrightarrow{u}A_1\xrightarrow{\Upsilon_1} A_2 \neq U_i \xrightarrow{u} A_1 \xrightarrow{\Upsilon_2} A_2$$

in $\mathcal{C}$. This leads to the required result. ||

The following propositions are dual to the above propositions:

Proposition 3.27. If $\{U_i\}_{i\in I}$ is a family of cogenerators on a V-category, then it is a family of cofibered cogenerators over all objects of the category.

Proposition 3.28. Let $\mathcal{C}$ be a V-category with equalizers. Then a family $\{U_i\}_{i\in I}$ of cofibered cogenerators over an object S is a family of cogenerators for $\mathcal{C}$.

Now , we have seen $\mathcal{C}_S \approx \mathcal{C} \approx \mathcal{C}^{S'}$ for a terminal objects S and an initial object S' by proposition 2.1. This will imply that the following proposition holds true for a terminal / an initial object.

Proposition 3.29. $\{U_i\}_{i \in I}$ is a family of generators/cogenerators in a category $\mathcal{C}$ if and only if it is a family of cofibered generators/ fibered cogenerators over a terminal/an initial object of $\mathcal{C}$.

(The proof can be easily checked).

Now , we obtain a characterization for fibered generators over an object.

Theorem 3.19. Let $\mathcal{C}$ be a V-category with coproducts. Then $\{U_i\}_{i \in I}$ is a family of fibered generators over an object S of $\mathcal{C}$ if and only if the object $\bigoplus_{i \in I} U_i$ is a fibered generator of $\mathcal{C}$ over S.

Proof. Let $\{U_i\}_{i \in I}$ be a family of fibered generators over S ==> there exist morphisms $u_i : U_i \longrightarrow S$, $\forall i$ .Since $\bigoplus_{i \in I} U_i$ is aCocoproduct of the given family , there exists a unique morphism $\Upsilon : \bigoplus U_i \longrightarrow S$ such that

$$U_i \xrightarrow{\mu_i} \bigoplus U_i \xrightarrow{\Upsilon} S = U_i \xrightarrow{u_i} S ,$$

where μ_i ,$i \in I$,are canonical injections. Next , let $\gamma_1 \neq \gamma_2 : (A_1,\alpha_1) \longrightarrow (A_2,\alpha_2)$ be two morphisms in $\mathcal{C}_S$. Therefore , $\gamma_1 \neq \gamma_2 : A_1 \longrightarrow A_2$ in $\mathcal{C}$ $\Longrightarrow$ there exists a morphism $u : U_i \longrightarrow A_1$ for some i such that $\gamma_1 u \neq \gamma_2 u$. Now consider a family $\{\beta_j : (U_i,u_i) \longrightarrow (A_1,\alpha_1)\}$ in $\mathcal{C}_S$ and hence $\{\beta_j : U_i \longrightarrow A_1\}$ in $\mathcal{C}$ such that $\beta_j = u$ for $j = i$, and for $j \neq i$ β_j exist , because $\mathcal{C}$ is a V-category. This implies that there exists a unique morphism $v : \oplus U_i \longrightarrow A_1$ such that

$$U_i \xrightarrow{\mu_i} \oplus U_i \xrightarrow{v} A_1 = U_i \xrightarrow{\beta_i} A_1 , \forall i ,$$

and since $\beta_i \in \mathcal{C}_S$ and by definition of coproduct $\alpha_1 v = \gamma$, hence $v \in \mathcal{C}_S$. Now, if , $\gamma_1 v = \gamma_2 v \Longrightarrow \gamma_1 v \mu_i = \gamma_2 v \mu_i$, $\forall i$ $\Longrightarrow \gamma_1 \beta_i = \gamma_2 \beta_i$ in particular , $\gamma_1 u = \gamma_2 u$, which is a contradiction $\Longrightarrow \gamma_1 v \neq \gamma_2 v$. Hence U is fibered generator over S.

Conversely , let $\oplus U_i$ be a fibered generator over S. This implies that there exists a morphism $\mu : \oplus U_i \longrightarrow S$ and hence $u_i : U_i \longrightarrow S = U_i \xrightarrow{\mu_i} \oplus U_i \xrightarrow{\mu} S$, $\forall i$.

Now, if $\gamma_1 \neq \gamma_2 : (A_1, \alpha_1) \longrightarrow (A_2, \alpha_2)$ be two morphisms in $\mathcal{C}_S$, then there exists a morphism $v : (\oplus U_i, \mu) \longrightarrow (A_1, \alpha_1)$ such that $\gamma_1 v \neq \gamma_2 v \Longrightarrow$ there exist at least one $i \in I$ such that $\gamma_1 v \mu_i \neq \gamma_2 v \mu_i$. Therefore, consider U_i with morphism $v\mu_i : (U_i, u_i) \longrightarrow (A_1, \alpha_1)$, which satisfies the condition (ii) of definition 3.10.

Dually, we have the following theorem :

Theorem 3.20. Let $\mathcal{C}$ be a V-category with products. Then $\{U_i\}_{i \in I}$ is a family of cofibered cogenerators if and only if the object $\prod_{i \in I} U_i$ (product of U_i's) is cofibered cogenerator over S.

The following theorems can be easily proved proceeding along the lines of the proofs of Theorem 3.4 and 3.5.

Theorem 3.21. Let $\mathcal{C}$ be a category with products and $S = \prod S_i$, a product of $\{S_i\}_{i \in I}$. Then an object U of $\mathcal{C}$ is fibered generator/cogenerator over S if and only if U is fibered generator/cogenerator over each S_i, $\forall\, i \in I$.

Theorem 3.22. Let $\mathcal{C}$ be a category with coproducts and $S = \bigoplus_{i \in I} S_i$ a coproduct, then U is cofibered generator/cogenerator over S if and only if U is cofibered generator/ cogenerator over each S_i, $\forall\, i \in I$.

3.6. Fibered and cofibered reflection and coreflection

In this section , we define fibered and cofibered reflective and coreflective subcategories over an object ,and show that a fibered/cofibered reflective/coreflective sub-category over any object is reflective/coreflective subcategory. The converse holds true only if the subcategory is a subcategory of a V-category. Also , if a subcategory is cofibered/fibered reflective/coreflective subcategory over an initial/a terminal object , then it is reflective/coreflective subcategory. Further , if $\mathcal{C}'$ is a reflective subcategory of a V-category, then $\mathcal{C}'$ is cofibered reflective subcategory of a V-category over each object of $\mathcal{C}'$.

Definition 3.11. Let $\mathcal{C}'$ be a subcategory of a category and S be an object of $\mathcal{C}$. Then we say that an object A of $\mathcal{C}$ has a fibered/cofibered reflection over S in $\mathcal{C}'$ if (i) there exists a morphism $\alpha : A \longrightarrow S$ / $\alpha : S \longrightarrow A$, (ii) the object $(A,\alpha)/(\alpha,A)$ of $\mathcal{C}_S$ / $\mathcal{C}^S$ has a reflection in ihe subcategory $\mathcal{C}'_S$ / $\mathcal{C}'^S$ (That is , there exists a object $(R(A),\gamma)$ in $\mathcal{C}'_S$ with a morphism $\rho_A :(R'A),\gamma) \longrightarrow (A,\alpha)$ in $\mathcal{C}_S$ such that for every object (A',α') in $\mathcal{C}'_S$ with morphism

$\beta : (A',\alpha') \longrightarrow (A,\alpha)$, there exists a unique morphism $\delta : (A',\alpha') \longrightarrow (R(A),\gamma)$ in $\mathcal{C}'_S$ such that the following diagram

$$\begin{tikzcd}
(A',\alpha') \arrow[d, "\delta"'] \arrow[dr, "\beta"] & \\
(R(A),\gamma) \arrow[r, "\rho_A"] & (A,\alpha)
\end{tikzcd}$$

is commutative.

The object $R(A)$ in $\mathcal{C}'$ is called fibered reflection object of A over S and the morphism ρ_A is called fibered reflection morphism of A over S.

Definition 3.12. Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$ and S be an object of $\mathcal{C}$. Then we say that an object A of $\mathcal{C}$ has fibered/cofibered coreflection in $\mathcal{C}'$ over S if there exists a morphism $\alpha : A \longrightarrow S$ / $\alpha : S \longrightarrow A$ in $\mathcal{C}$ and $(A,\alpha)/(\alpha,A)$ in $\mathcal{C}_S$ / $\mathcal{C}^S$ has coreflection in the subcategory in $\mathcal{C}'_S$/ $\mathcal{C}'^S$.

Definition 3.13. If each object of $\mathcal{C}$ has fibered/cofibered reflection in $\mathcal{C}'$ over an object S of $\mathcal{C}$, then we say that

$\mathcal{C}'$ is fibered/cofibered reflective subcategory of $\mathcal{C}$ over S.

Definition 3.14. If each object of $\mathcal{C}$ has a fibered/cofibered coreflection in $\mathcal{C}'$ over an object S of $\mathcal{C}$, then we say that $\mathcal{C}'$ is fibered/cofibered coreflective subcategory of $\mathcal{C}$ over S.

Proposition 3.33. If $\mathcal{C}'$ is a fibered reflective subcategory of $\mathcal{C}$ over an object S, then $\mathcal{C}'$ is a reflective subcategory of $\mathcal{C}$.

Proof. Let $A \in \mathcal{C}$. Since $\mathcal{C}'$ is fibered reflective subcategory over S, there exists a morphism $\alpha : A \longrightarrow S$ and an object $(R(A),\gamma)$ in $\mathcal{C}'_S$ with a morphism

$$\rho_A : (R(A),\gamma) \longrightarrow (A,\alpha) = \begin{tikzcd} R(A) \arrow[rr, "\rho_A"] \arrow[dr, "\gamma"'] & & A \arrow[dl, "\alpha"] \\ & S & \end{tikzcd}$$

Thus, we have an object R(A) in $\mathcal{C}'$ with a morphism $\rho_A : R(A) \longrightarrow A$.

Next, if A' be an object in $\mathcal{C}'$ with a morphism $\xi : A' \longrightarrow A$, then we have $\alpha'(\text{say}) = \alpha\xi : A' \longrightarrow S$ and a morphism $\xi : (A',\alpha') \longrightarrow (A,\alpha)$. Therefore , by definition 3.12, there exists a unique morphism $\delta : (A',\alpha') \longrightarrow (R(A),\Upsilon)$ in $\mathcal{C}'_S$ such that the following diagram is commutative :

$$\begin{tikzcd}
(A',\alpha') \arrow[d, "\delta"'] \arrow[dr, "\xi"] & \\
(R(A),\Upsilon) \arrow[r, "\rho_A"] & (A,\alpha)
\end{tikzcd}$$

Hence , we have a unique $\delta : A' \longrightarrow R(A)$ in $\mathcal{C}'$ such that the following diagram

$$\begin{tikzcd}
A' \arrow[d, "\delta"'] \arrow[dr, "\xi"] & \\
R(A) \arrow[r, "\rho_A"] & A
\end{tikzcd}$$

is commutative. Thus , $\mathcal{C}'$ is a reflective subcategory of $\mathcal{C}$. ||

Dually , we have the following proposition :

Proposition 3.34. If $\mathcal{C}'$ is cofibered càreflective subcategory of a category $\mathcal{C}$ over an object S of $\mathcal{C}$, then $\mathcal{C}'$ is coreflection subcategory of $\mathcal{C}$. ||

Proposition 3.35. If $\mathcal{C}'$ is cofibered reflective subcategory over an initial object S of $\mathcal{C}$, then $\mathcal{C}'$ is reflective subcategory of $\mathcal{C}$.

Proof. Suppose A is an object of $\mathcal{C}$. Since $\mathcal{C}'$ is cofibered reflective subcategory of $\mathcal{C}$, there exists a morphism $\alpha : S \longrightarrow A$ and an object $(\gamma, R(A))$ in $\mathcal{C}'^{S}$ with a morphism $\rho^{A} : (\gamma, R(A)) \longrightarrow (\alpha, A)$. Thus , we have an object R(A) in $\mathcal{C}'$ with a morphism $\rho^{A} : R(A) \longrightarrow A$.

Next , if A' be another object of $\mathcal{C}'$ with a morphism $\xi : A' \longrightarrow A$ and since S is initial object , there exists a morphism $\alpha' : S \longrightarrow A'$ such that the following diagram

$$\begin{tikzcd} & S \arrow[dl, "\alpha'"'] \arrow[dr, "\alpha"] & \\ A' \arrow[rr, "\xi"] & & A \end{tikzcd}$$

is commutative. Thus , $\xi : (\alpha', A') \longrightarrow (\alpha, A) \in \mathcal{C}^{S}$, and therefore , by definition 3.12 , there exists a unique morphism $\delta : (\alpha', A') \longrightarrow (\gamma, R(A))$ in $\mathcal{C}'^{S}$, such that the following diagram

$$\begin{tikzcd}
(\alpha', A') \arrow[d, "\delta"'] \arrow[drr, "\xi"] & & \\
(\gamma, R(A)) \arrow[rr, "\rho_A"] & & (\alpha, A)
\end{tikzcd}$$

is commutative.

Thus , there exists a unique morphism $\delta : A' \longrightarrow R(A)$ in $\mathcal{C}'$ such that the following diagram

$$\begin{tikzcd}
A' \arrow[d, "\delta"'] \arrow[drr, "\xi"] & & \\
R(A) \arrow[rr, "\rho_A"] & & A
\end{tikzcd}$$

is commutative. Thus , $\mathcal{C}'$ is reflective subcategory of $\mathcal{C}$. ||

Dually , we have the following :

Proposition 3.36. If $\mathcal{C}'$ is fibered coreflective subcategory of a category $\mathcal{C}$ over a terminal object S of $\mathcal{C}$. Then $\mathcal{C}'$ is coreflective subcategory of $\mathcal{C}$. ||

Proposition 3.37. Let $\mathcal{C}$ be a V-category. Then a reflective subcategory $\mathcal{C}'$ of $\mathcal{C}$ is fibered reflective subcategory of over all objects of $\mathcal{C}$.

Proof. Let A be an object of $\mathcal{C}$. Since $\mathcal{C}$ is a V- category there exists a morphism $\alpha : A \longrightarrow S$. Moreover, since $\mathcal{C}$ is reflective subcategory of $\mathcal{C}$, there exists an object $R(A)$ in $\mathcal{C}'$ with a morphism $\rho_A : R(A) \longrightarrow A$ in $\mathcal{C}'$. Now, consider $\gamma = \alpha \;\; \rho_A : R(A) \longrightarrow S$, then we have an object $(R(A),\gamma)$ in $\mathcal{C}'_S$ with a morphism $\rho_A : (R(A),\gamma) \longrightarrow (A,\alpha)$ in $\mathcal{C}'_S$.
Next, if (A',α') is another object in $\mathcal{C}'_S$ with a morphism $\xi : (A',\alpha') \longrightarrow (A,\alpha)$, then we have object A' in $\mathcal{C}'$ with a morphism $\xi : A' \longrightarrow A$. Now, since $\rho_A : R(A) \longrightarrow A$ is reflection of A in $\mathcal{C}'$, there exists a unique morphism $\delta : A' \longrightarrow R(A)$ such that

$$A' \xrightarrow{\delta} R(A) \xrightarrow{\rho_A} A = A' \xrightarrow{\xi} A .$$

Next,

$$\gamma\delta = \alpha \;\; \rho_A\delta = \alpha\,\xi = \alpha'$$

$\Longrightarrow \gamma : (A',\alpha') \longrightarrow (R(A),\gamma)$ ~~in~~ in $\mathcal{C}_S$. Thus we have an unique morphism $\delta : (A',\alpha') \longrightarrow (R(A),\gamma)$ such that

$$(A',\alpha') \xrightarrow{\delta} (R(A),\gamma) \xrightarrow{\rho_A} (A,\alpha) = (A',\alpha') \xrightarrow{\xi} (A,\alpha)$$

Hence the result follows. $\|$

Dually , we have the following proposition :

Proposition 3.38. Let $\mathcal{C}$ be a V-category and $\mathcal{C}'$ be a coreflection subcategory of $\mathcal{C}$. Then $\mathcal{C}$ is cofibered coreflection subcategory of $\mathcal{C}$ over all objects of $\mathcal{C}$. $\|$

Proposition 3.39. Let $\mathcal{C}$ be a V-category and $\mathcal{C}'$ is reflection subcategory of $\mathcal{C}$. Then $\mathcal{C}'$ is cofibered reflective subcategory over all objects of $\mathcal{C}'$.

Proof. Let S be an object of $\mathcal{C}'$ and A be an object of $\mathcal{C}$. Since $\mathcal{C}'$ is reflective subcategory of $\mathcal{C}$, we have an object R(A) in $\mathcal{C}$ with a morphism $\rho_A : R(A) \longrightarrow A$. Now , since $\mathcal{C}$ is a V-category , there exists a morphism $\beta : S \longrightarrow R(A)$. Next , considering $\alpha = \rho_A \beta : S \longrightarrow A$, we have an object $(\beta, R(A))$ in $\mathcal{C}'^{S}$ with a morphism $\rho_A : (\beta, R(A)) \longrightarrow (\alpha, A)$. Furthermore , if (α', A') be another object in $\mathcal{C}'^{S}$ with a morphism $\xi : (\alpha', A') \longrightarrow (\alpha, A)$, then we have an object A' in $\mathcal{C}'$ with a morphism $\xi : A' \longrightarrow A$ in $\mathcal{C}$ and therefore , there exists a unique morphism $\gamma : A' \longrightarrow R(A)$ in $\mathcal{C}'$ such that $A' \xrightarrow{\gamma} R(A) \xrightarrow{\rho_A} A = A' \xrightarrow{\xi} A$. Thus , we have the following commutative diagram

$$\begin{tikzcd}
 & A' \arrow[d, "\gamma"] \arrow[drr, "\xi"] & & \\
 & R(A) \arrow[rr, "\rho_A"] & & A \\
S \arrow[uur, "\alpha'"] \arrow[ur, "\beta"] \arrow[urrr, "\alpha"'] & & &
\end{tikzcd}$$

giving $\rho_A \gamma \alpha' = \xi\alpha' = \alpha = \rho_A \beta$.

Moreover , $S \in \mathcal{C}'$ and ρ_A is reflection morphism from $R(A) \longrightarrow A$.

Therefore , $\gamma \alpha' = \beta \Longrightarrow$

$\gamma : (\alpha', A') \longrightarrow (\beta, R(A))$ belongs to $\mathcal{C}'_S$ which is unique and satisfies

$$(\alpha', A') \xrightarrow{\gamma} (\beta, R(A)) \xrightarrow{\rho_A} (\alpha, A) = (\alpha', A') \xrightarrow{\xi} (\alpha, A)$$

Thus result follows. ||

Dually , we have the following proposition.

Proposition 3.40. Let $\mathcal{C}$ be a V-category and $\mathcal{C}'$ is a coreflective subcategory . Then $\mathcal{C}'$ is fibered coreflective subcategory of $\mathcal{C}$ over all objects of $\mathcal{C}'$.

The following proposition can be easily checked using the definition of initial and terminal objects and propositions 3.33 and 3.36.

Proposition 3.41. If $\mathcal{C}'$ is reflective/ coreflective subcategory of $\mathcal{C}$ if and only if $\mathcal{C}'$ is cofibered/fibered reflective/coreflective subcategory over an initial / a terminal object, or if and only if $\mathcal{C}'$ is fibered/ cofibered reflective/ coreflective subcategory of $\mathcal{C}$ over a terminal/ an initial object.

References

Bucure [1] , Freyd [3] , Mitchell [13] , Pareigis [14] .

CHAPTER IV

FIBERED FACTORIZATIONS

4.0. <u>Introduction</u>. Isbell [6] introduced the concept of extremal epimorphism that ' an epimorphism $e : A \longrightarrow B$ is <u>extremal</u> (i) if $A \xrightarrow{e} B = A \xrightarrow{\alpha} B \xrightarrow{e'} B$, i.e. $e = e'\alpha$, and (ii) if e' is a monomorphism, then e' is an isomorphism! Dually, a monomorphism m is extremal, if $m = \beta m'$ and m' is an epimorphism implies that m' is an isomorphism. Herrilich, H. [5], very recently, generalized it to an <u>extremal epi-mono factorization</u> of a morphism. A morphism f in a category is said to have an extremal epi-mono factorization if for some extremal epimorphism e in $\mathcal{C}$ and some monomorphism m in $\mathcal{C}$, $f = me$. We have similarly an epi-extremal mono factorization.

Further, Kelly, G.M. [10] introduced the concept of regular and strong epimorphisms as follows :

A morphism $f : A \longrightarrow B$ in a category $\mathcal{C}$ is said to be a <u>regular epimorphism</u> if any $g : A \longrightarrow C$ in $\mathcal{C}$, for some $x,y : K \longrightarrow A$ satisfying,

$$gx = gy \text{ , whenever } fx = fy \text{ ,}$$

has a decomposition $g = hf$ for a unique morphism $h : B \longrightarrow C$

in $\mathcal{C}$, i.e., f is a coequalizer in a general sense. Also, an epimorphism e is said to be <u>strong</u> if , whenever ve = mu with m a monomorphism , there exists a unique morphism w such that the following diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] & B \arrow[d, "v"] \arrow[dl, dashed, "w"'] \\ C \arrow[r, "m"] & D \end{tikzcd}$$

is commutative.

We can similarly , define regular and strong monomorphisms. Kelly observed that regular epimorphisms/monomorphisms are strong epimorphisms/monomorphisms and strong epis/monos are extremal epi/mono. Also , he introduced the following factorizations :

(i) If a morphism f has a factorization f = me , where e is regular epimorphism and where ex = ey , whenever fx = fy, then me is called <u>the regular factorization of f</u> .

(ii) If a morphism f has a factorization f = mp , where p is strong epimorphism and m is a monomorphism , then mp is called a <u>canonical factorization of f</u>.

In this chapter , we define fibered/cofibered regular (Definitions 4.1/4.2) strong (Definitions 4.5/4.6) and extremal (Definitions 4.9/4.10) epimorphisms and monomorphisms, and obtain more or less similar properties as in [10] . We also define and study fibered/cofibered regular , canonical and extremal epi-mono factorizations.

4.1. <u>Fibered and cofibered regular epimorphisms and monomorphisms</u>

In this section , we define fibered and cofibered regular epimorphisms and monomorphisms over some object in a category and observe that in V-categories a fibered/cofibered regular epimorphism/monomorphism over any object is a regular epimorphism / monomorphism ; and also , over terminal and initial objects , both the terms ' fibered regular epimorphisms' and 'regular epimorphisms' coincides. Also we obtain some properties like those of composition , left cancellation etc., of regular morphisms.

Definition 4.1. An epimorphism $e : A \longrightarrow B$ in $\mathcal{C}$ is called fibered/cofibered regular epimorphism over an object S of $\mathcal{C}$ if $f \in \mathcal{C}_S / \mathcal{C}^S$ and is regular epimorphism in $\mathcal{C}_S / \mathcal{C}^S$.

That is , in fibered case , there exist morphisms $\alpha : A \longrightarrow S$, $\beta : B \longrightarrow S$ such that $A \xrightarrow{e} B \xrightarrow{\beta} S = A \xrightarrow{\alpha} S$, and that any $g : (A,\alpha) \longrightarrow (C,\gamma)$ in $\mathcal{C}_S$, satisfying $gx = gy$, whenever $ex = ey$ for any pair of morphisms $x,y : (K,\gamma) \longrightarrow (A,\alpha)$ in $\mathcal{C}_S$, has a decomposition $g = he$, for a unique morphism h in $\mathcal{C}_S$.

Dually :

Definition 4.2. A monomorphism $m : A \longrightarrow B$ in $\mathcal{C}$ is called fibered/ cofibered regular monomorphism over an object S of if $m \in \mathcal{C}_S / \mathcal{C}^S$ and is regular monomorphism in $\mathcal{C}_S / \mathcal{C}^S$.

That is , in fibered case , there exist morphisms $\alpha : A \longrightarrow S$, $\beta : B \longrightarrow S$ such that $A \xrightarrow{m} B \xrightarrow{\beta} S = A \xrightarrow{\alpha} S$, and that any $g : (C,\gamma) \longrightarrow (B,\beta)$ satisfying $xg = yg$, whenever $xm = ym$, for any pair of morphisms x , y in $\mathcal{C}_S$, has a unique decomposition $g = mh$ in $\mathcal{C}_S$.

Proposition 4.1. Let $\mathcal{C}$ be a V-category. Then a regular epimorphism is fibered regular epimorphism over any object S.

Proof. Let $e : A \longrightarrow B$ be a regular epimorphism. Since $\mathcal{C}$ is a V-category, there exists a morphism $\beta : B \longrightarrow S$, then consider $\alpha = \beta f : A \longrightarrow S$ and therefore, $A \xrightarrow{\alpha} S = A \xrightarrow{e} B \xrightarrow{\beta} S$.

Next , since e is epi in $\mathcal{C}$, by lemma 1.19, e is epi in $\mathcal{C}_S$.
Now suppose for x,y : $(K,k) \longrightarrow (A,\alpha)$

$$(K,k) \xrightarrow{x} (A,\alpha) \xrightarrow{g} (C,\gamma) = (K,k) \xrightarrow{y} (A,\alpha) \xrightarrow{g} (C,\gamma),$$

whenever

$$(K,k) \xrightarrow{x} (A,\alpha) \xrightarrow{e} (B,\beta) = (K,k) \xrightarrow{y} (A,\alpha) \xrightarrow{e} (B,\beta)$$

Now , if for any x' , y' : $K' \longrightarrow A$

$$K' \xrightarrow{x'} A \xrightarrow{e} B = K' \xrightarrow{y'} A \xrightarrow{e} B$$

then , considering

$$\begin{aligned} k' : K' \longrightarrow S = K' \xrightarrow{x'} A \xrightarrow{\alpha} S &= K' \xrightarrow{x'} A \xrightarrow{e} B \xrightarrow{\beta} S \\ &= K' \xrightarrow{y'} A \xrightarrow{e} B \xrightarrow{\beta} S \\ &= K' \xrightarrow{y'} A \xrightarrow{\alpha} S , \end{aligned}$$

$$(K',k') \xrightarrow{x'} (A,\alpha) \xrightarrow{e} (B,\beta) = (K',k') \xrightarrow{y'} (A,\alpha) \xrightarrow{e} (B,\beta)$$

$$\Longrightarrow (K',k') \xrightarrow{x'} (A,\alpha) \xrightarrow{g} (C,\gamma) = (K',k') \xrightarrow{y'} (A,\alpha) \xrightarrow{g} (C,\gamma)$$

in $\mathcal{C}_S \Longrightarrow gx' = gy'$ in $\mathcal{C}$, i.e., $gx' = gy'$ whenever

$ex' = ey'$ in $\mathcal{C}$. Hence, by definition of regular epimorphism, there exists a unique morphism $h : B \longrightarrow C$ in $\mathcal{C}$ such that

$$A \xrightarrow{e} B \xrightarrow{h} C = A \xrightarrow{g} C$$

Next, $A \xrightarrow{e} B \xrightarrow{h} C \xrightarrow{\gamma} S = A \xrightarrow{g} C \xrightarrow{\gamma} S$

$= A \xrightarrow{\alpha} S = A \xrightarrow{e} B \xrightarrow{\beta} S$. Now, since e is epi,

$B \xrightarrow{h} C \xrightarrow{\gamma} S = B \xrightarrow{\beta} S$, i.e. $h : (B,\beta) \longrightarrow (C,\gamma) \in \mathcal{C}_S$, such that

$$(A,\alpha) \xrightarrow{e} (B,\beta) \xrightarrow{h} (C,\gamma) = (A,\alpha) \xrightarrow{g} (C,\gamma)$$

This proves the proposition. ||

Dually, we have the following proposition :

Proposition 4.2. Let $\mathcal{C}$ be a V-category and m is a regular monomorphism. Then m is cofibered regular monomorphism over any object S of $\mathcal{C}$.

Proposition 4.3. An epimorphism $e : A \longrightarrow B$ in a category $\mathcal{C}$ is regular epimorphism if and only if it is fibered/cofibered regular epimorphism over a terminal/initial object.

Proof. Let $e : A \longrightarrow B$ be a regular epimorphism and S be a terminal object. Then there exist unique morphisms $\alpha : A \longrightarrow S$ and $\beta : B \longrightarrow S$ and, since S is terminal object,

$A \xrightarrow{e} B \xrightarrow{\beta} S = A \xrightarrow{\alpha} S$. Now, to prove that e is fibered regular epimorphism proceed as we have done in proposition 4.1.

For the converse, let $e : A \longrightarrow B$ be a fibered regular epimorphism over a terminal object $S \Longrightarrow$ there exist $\alpha : A \longrightarrow S$ and $\beta : B \longrightarrow S$ such that $A \xrightarrow{\alpha} S = A \xrightarrow{e} B \xrightarrow{\beta} S$.

Next, suppose for any $x, y : K \longrightarrow A$ in $\mathcal{C}$

$$K \xrightarrow{x} A \xrightarrow{g} C = K \xrightarrow{y} A \xrightarrow{g} C$$

whenever

$$K \xrightarrow{x} A \xrightarrow{e} B = K \xrightarrow{y} A \xrightarrow{e} B .$$

Now, as S is a terminal object, we have

$$\begin{tikzcd} K \arrow[r, "x"] \arrow[dr, "k"'] & A \arrow[r, "g"] \arrow[d, "\alpha"] & C \arrow[dl, "\gamma"] \\ & S & \end{tikzcd} = \begin{tikzcd} K \arrow[r, "y"] \arrow[dr, "k"'] & A \arrow[r, "g"] \arrow[d, "\alpha"] & C \arrow[dl, "\gamma"] \\ & S & \end{tikzcd}$$

i.e. $(K,k) \xrightarrow{x} (A,\alpha) \xrightarrow{g} (C,\gamma) = (K,k) \xrightarrow{y} (A,\alpha) \xrightarrow{g} (C,\gamma)$

whenever

$$(K,k) \xrightarrow{x} (A,\alpha) \xrightarrow{e} (B,\beta) = (K,k) \xrightarrow{y} (A,\alpha) \xrightarrow{e} (B,\beta).$$

Therefore , since e is fibered regular epi over S , there exists a unique morphism $h : (B,\beta) \longrightarrow (C,\gamma)$ in $\mathcal{C}_S$, such that

$$(A,\alpha) \xrightarrow{e} (B,\beta) \xrightarrow{h} (C,\gamma) = (A,\alpha) \xrightarrow{g} (C,\gamma).$$

And, hence we have a unique $h \in \mathcal{C}$ such that

$$A \xrightarrow{e} B \xrightarrow{h} C = A \xrightarrow{g} C$$

Dually , we have

Proposition 4.4. A monomorphism $m : A \longrightarrow B$ in a category is a regular monomorphism if and only if m is fibered/cofibered regular monomorphism over some terminal/initial object of $\mathcal{C}$.

Lemma 4.1. If a morphism f belongs to $\mathcal{C}_S$, then retraction of f also belongs to $\mathcal{C}_S$.

Proof. Let $f : (A,\alpha) \longrightarrow (B,\beta) \in \mathcal{C}_S$. That is , we have the following commutatice diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\ & S & \end{tikzcd}$$

and let Υ be retraction of f. That is , $\Upsilon : B \longrightarrow A$ such that $f\Upsilon = I_B$.

Now, $\beta f \Upsilon = \alpha\Upsilon$

$\Longrightarrow \beta I_B = \alpha\Upsilon$

$\Longrightarrow \beta = \alpha\Upsilon$, i.e. $B \xrightarrow{\Upsilon} A \xrightarrow{\alpha} S = B \xrightarrow{\beta} S \Longrightarrow$

$\Upsilon : (B,\beta) \longrightarrow (A,\alpha) \in \mathcal{C}_S$.

Dually , we have the following lemma :

Lemma 4.2. If f belongs to $\mathcal{C}^S$, then contraction of ~ 3o belongs to $\mathcal{C}$.

Kelly, G.M. [10] proved that fg is regular epimorphism if f is a regular epimorphism and g is a retraction. We have the similar result for fibered regular epimorphisms over an object S.

Proposition 4.5. If e is fibered regular epimorphism over S and Υ is a retraction of f in $\mathcal{C}$, then $e\Upsilon$ is fibered regular epimorphism over S.

Proof. Since $e : A \longrightarrow B$ is fibered regular epimorphism over S, there exist morphisms $\alpha : A \longrightarrow S$ and $\beta : B \longrightarrow S$ such that

$$A \xrightarrow{\alpha} S = A \xrightarrow{e} B \xrightarrow{\beta} S.$$

Let $\gamma : C \longrightarrow A$ be a morphism in $\mathcal{C}$, then $g = \alpha\gamma : C \xrightarrow{\gamma} A \xrightarrow{\alpha} S$ $= C \xrightarrow{\gamma} A \xrightarrow{e} B \xrightarrow{\beta} S \implies e\gamma : (C,g) \longrightarrow (B,\beta) \in \mathcal{C}_S$ and, since e is regular epimorphism in $\mathcal{C}_S$ and, by lemma 4.1, γ is retraction belongs to $\mathcal{C}_S$. Therefore, by result of Kelly stated above, $e\gamma$ is regular epimorphism in $\mathcal{C}_S$. i.e. $e\gamma$ is fibered regular epimorphism in $\mathcal{C}$ over S.

Dually, we have the following :

Proposition 4.6. If m is cofibered regular monomorphism over S and γ is coretraction in $\mathcal{C}$, then γm is cofibered regular monomorphism over S.

Proposition 4.7. Let $C \xrightarrow{g} B \xrightarrow{f} A$ be morphisms in and S be an object of $\mathcal{C}$ such that the following diagram

$$\begin{tikzcd}
 & S \arrow[dl, "\gamma"'] \arrow[d, "\beta"] \arrow[dr, "\alpha"] & \\
C \arrow[r, "g"] & B \arrow[r, "f"] & A
\end{tikzcd}$$

is commutative. Then fg is cofibered regular epimorphism over S if f is cofibered regular epimorphism over S and g is retraction in $\mathcal{C}^S$.

Proof. By hypothesis, fg belongs to $\mathcal{C}^S$ and as f is cofibered regular epimorphism over S, f is regular epimorphism in $\mathcal{C}^S$. Hence, by Kelly's result, fg is cofibered regular epimorphism over S.

Dually, we have the following :

Proposition 4.8. Let $C \xrightarrow{g} B \xrightarrow{f} A$ be morphisms in and S be an object of $\mathcal{C}$ such that the following diagram

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[dr, "\gamma"'] & B \arrow[r, "f"] \arrow[d, "\beta"] & A \arrow[dl, "\alpha"] \\
 & S &
\end{tikzcd}$$

is commutative. Then fg is fibered regular monomorphism over S if g is fibered regular monomorphism over S and f is coretraction in $\mathcal{C}_S$.

Kelly [10] proved the following result about right divisibility of regular morphism :

Proposition 4.9. If fg is a regular epimorphism and if g is an epimorphism , f is a regular epimorphism.

Now , we have the following results about the right divisibility of fibered regular morphisms:

Proposition 4.10. Let g be an epimorphism in $\mathcal{C}$, fg be fibered regular epimorphism over an object S. Then f is fibered regular epimorphism over S.

Proof. Since $C \xrightarrow{g} B \xrightarrow{f} A$ is fibered regular epimorphism over S , there exist morphisms $\gamma : C \longrightarrow S$ and $\alpha : A \longrightarrow S$ such that the following diagram

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[dr, "\gamma"'] & B \arrow[r, "f"] \arrow[d, "\alpha f"] & A \arrow[dl, "\alpha"] \\
& S &
\end{tikzcd}$$

is commutative.

Now , taking $\beta = \alpha f : B \longrightarrow S$, we have $f \in \mathcal{C}_S$ and since g is epimorphism in $\mathcal{C}$, by lemma 1.19 , it is epi in $\mathcal{C}_S$. Therefore , by proposition 4.9 , f is regular epimorphism in $\mathcal{C}_S$ and hence f is fibered regular epimorphism over S.

Similarly , we have the following proposition :

Proposition 4.11. If fg is cofibered regular epimorphism over S and g is epimorphism . Then f is cofibered regular epimorphism over S.

Now, we have the following propositions dual to 4.10 and 4.11.

Proposition 4.12. If fg is fibered regular monomorphism over S and f is monomorphism then g is fibered regular monomorphism over S.

Proposition 4.13. If fg is cofibered regular monomorphism over S and f is a monomorphism in $\mathcal{C}$, then g is cofibered regular monomorphism over S.

4.2. <u>Fibered and cofibered regular factorizations</u>

In this section , we define fibered and cofibered regular epi and mono factorizations over some object of a category and observe that if an epimorphism/ monomorphism has a regular epi/mono factorization , then it has fibered epi/ cofibered mono factorization and further obtain that if a morphism f has fibered regular factorization $n\Upsilon$ over some object S , then n is an isomorphism if and only if f is fibered regular epimorphism over S , and Υ is an isomorphism if and only if f is a monomorphism.

Definition 4.3. A morphism $f : A \longrightarrow B$ in a category $\mathcal{C}$ is said to have a fibered/cofibered regular epifactorization over an object S of $\mathcal{C}$ if f has regular epifactorization in $\mathcal{C}_S$ / $\mathcal{C}^S$. That is , in fibered case (i) there exist morphisms $\alpha : A \longrightarrow S$, $\beta : B \longrightarrow S$ such that $A \xrightarrow{f} B \xrightarrow{\beta} S = A \xrightarrow{\alpha} S$ and (ii) $f : (A,\alpha) \longrightarrow (B,\beta)$ has a factorization

$$(A,\alpha) \xrightarrow{f} (B,\beta) = (A,\alpha) \xrightarrow{\Upsilon} (A',\alpha') \xrightarrow{n} (B,\beta)$$,where Υ is a regular epimorphism in $\mathcal{C}_S$ and $\Upsilon s = \Upsilon f$ whenever $fs = ft$, for any $s,t : (K,k) \longrightarrow (A,\alpha)$.

Definition 4.4. A morphism $f : A \longrightarrow B$ in $\mathcal{C}$ has a fibered/ cofibered regular monofactorization over an object S if f has a regular monofactorization in $\mathcal{C}_S$ / $\mathcal{C}^S$. That is , in fibered case, (i) there exist morphisms $\alpha : A \longrightarrow S$, $\beta : B \longrightarrow S$

such that $A \xrightarrow{f} B \xrightarrow{\beta} S = A \xrightarrow{\alpha} S$ and (ii) f has factorization $(A,\alpha) \xrightarrow{f} (B,\beta) = (A,\alpha) \xrightarrow{e} (A',\alpha') \xrightarrow{q} (B,\beta)$, where q is regular monomorphism in $\mathcal{C}_S$ and $sq = fq$ whenever $sf = tf$ for any pair of morphisms $s,t : (B,\beta) \longrightarrow (K,k)$.

Proposition 4.14. Let $\mathcal{C}$ be a V-category and a morphism $f : A \longrightarrow B$ in $\mathcal{C}$ has a regular epifactorization. Then f has fibered regular epifactorization over any object S in $\mathcal{C}$.

Proof. Since $\mathcal{C}$ is a V-category, there exists $\beta : B \longrightarrow S$. Considering $\alpha = \beta f : A \longrightarrow S$, we have $A \xrightarrow{f} B \xrightarrow{\beta} S = A \xrightarrow{\alpha} S$. Next, let $A \xrightarrow{f} B = A \xrightarrow{\gamma} A' \xrightarrow{m} B$ be regular epifactorization of f in $\mathcal{C}$. Then, defining $\alpha' = A' \longrightarrow S = A' \xrightarrow{m} B \xrightarrow{\beta} S$, we have a factorization

$$(A,\alpha) \xrightarrow{f} (B,\beta) = (A,\alpha) \xrightarrow{\gamma} (A',\alpha') \xrightarrow{m} (B,\beta)$$

in $\mathcal{C}_S$. Now, since γ is regular, it is fibered regular over S by proposition 4.1. Next, if

$$(K,k) \xrightarrow{s} (A,\alpha) \xrightarrow{f} (B,\beta) = (K,k) \xrightarrow{t} (A,\alpha) \xrightarrow{f} (B,\beta)$$

$\Longrightarrow fs = ft$ in $\mathcal{C}$ $\Longrightarrow \gamma s = \gamma f$ in $\mathcal{C}$, hence in $\mathcal{C}_S$.

Dually , we have the following proposition :

Proposition 4.15. Let $\mathcal{C}$ be a V-category and a morphism $f : A \longrightarrow B$ in $\mathcal{C}$ has regular monofactorization. Then f has cofibered regular monofactorization over any object S.||

Using proposition 4.3 and definition of terminal and initial objects , we have the following :

Theorem 4.1. A morphism $f : A \longrightarrow B$ in $\mathcal{C}$ has regular epifactorization if and only if f has fibered/cofibered regular epifactorization over some terminal/initial object.

Dually , we have the following :

Theorem 4.2. A morphism $f : A \longrightarrow B$ in $\mathcal{C}$ has regular monofactorization if and only if f has fibered/cofibered regular monofactorization over some terminal/initial object.

Lemma 4.3. A morphism $f : (A,\alpha) \longrightarrow (B,\beta)$ in $\mathcal{C}_S$ is an isomorphism in $\mathcal{C}_S$ if and only if it is an isomorphism in $\mathcal{C}$.

Proof. Let $f : (A,\alpha) \longrightarrow (B,\beta)$. That is , $f : A \longrightarrow B$. Suppose $f : A \longrightarrow B$ is an isomorphism in $\mathcal{C}$ $\Longrightarrow$ there exist a $g : B \longrightarrow A$ such that $f\,g = I_B$ and $g\,f = I_A$.

Now,

$$B \xrightarrow{\beta} S = B \xrightarrow{g} A \xrightarrow{f} B \xrightarrow{\beta} S = B \xrightarrow{g} A \xrightarrow{\alpha} S$$

$\Longrightarrow \quad g : (B,\beta) \longrightarrow (A,\alpha)$

such that $gf = I_{(A,\alpha)}$, $fg = I_{(B,\beta)}$.

$\Longrightarrow f \in \mathcal{C}_S$ is isomorphism.

The converse is obvious.

Kelly [10] proved the following proposition :

Proposition 4.16. If f has the regular factorization $m\Upsilon$, then m is an isomorphism if and only if f is a regular epimorphism , and Υ is an isomorphism if and only if it is a monomorphism.

Now with the help of Lemma 4.3 and the above proposition 4.16 , we prove the following proposition :

Proposition 4.17. If f has fibered regular epifactorization $m\Upsilon$ over S , then m is an isomorphism if and only if f is fibered regular epimorphism over S , and Υ is an isomorphism if and only if f is a monomorphism.

Proof. Suppose m is an isomorphism. Then, by lemma 4.3 , m is an isomorphism in $\mathcal{C}_S$, and , since Υ is regular epimorphism in $\mathcal{C}_S$, then by proposition 4.16 , f is regular epimorphism in $\mathcal{C}_S$. That is , f is fibered regular epimorphism over S.

Conversely, let f be fibered regular epimorphism over S ==> f is a regular epimorphism in $\mathcal{C}_S$. Then , by proposition 4.16 , m is an isomorphism in $\mathcal{C}_S$ hence m is isomorphism in $\mathcal{C}$ by lemma 4.3.

The other part can also be proved similarly.

Dually , we have

Proposition 4.18. If $f : A \longrightarrow B$ has cofibered regular monofactorization over S an qm , then m is an isomorphism if and only if f is fibered regular mono over S , and q is an isomorphism if and only if f is an epimorphism in $\mathcal{C}^S$.

The following proposition follows directly by definition and proposition 2.6 [10] .

Proposition 4.19. Let

$$\begin{array}{ccc} (A,\alpha) & \xrightarrow{\quad f \quad} & (B,\beta) \\ \Big\downarrow a & & \Big\downarrow b \\ (C,\phi) & \xrightarrow{\quad g \quad} & (D,\delta) \end{array}$$

be a commutative diagram in $\mathcal{C}_S$ such that f and g has fibered regular epifactorization over S as in the following diagram

$$\begin{tikzcd}
(A,\alpha) \arrow[r, "\gamma"] \arrow[d, "a"] & (A',\alpha') \arrow[r, "m"] \arrow[d, "w"] & (B,\beta) \arrow[d, "b"] \\
(C,\phi) \arrow[r, "\gamma'"] & (C',\phi') \arrow[r, "m'"] & (D,\delta)
\end{tikzcd}$$

Then there exists a unique morphism w in $\mathcal{C}_S$ sending the diagram commutative.

4.3. Fibered and cofibered strong epimorphismsand monomorphisms

In this section , we define fibered and cofibered strong epimorphisms and monomorphisms over an object. We find that if an epimorphism/monomorphism f : A ——> B is cofibered/fibered strong epimorphism/monomorphism over ah object S, then f is strong epimorphism/ monomorphism , and in a V-category , every strong epimorphism/monomorphism is fibered/cofibered strong epimorphism/monomorphism. We also establish that a fibered regular epimorphism/monomorphism over an object S is fibered strong epimorphism/monomorphism over S. Besides this,we give certain other properties regarding strong morphisms.

Definition 4.5. We call an epimorphism $e : A \longrightarrow B$ in $\mathcal{C}$ a fibered/cofibered strong epimorphism over an object S if it is strong epimorphism in $\mathcal{C}_S$ / $\mathcal{C}^S$. That is, in the fibered case (i) there exist morphisms $\alpha : A \longrightarrow S$, $\beta : B \longrightarrow S$ such that $A \xrightarrow{f} B \xrightarrow{\beta} S = A \xrightarrow{\alpha} S$ and (ii) if we have the following commutative diagram with a monomorphism m in $\mathcal{C}_S$,

$$\begin{tikzcd}
(A,\alpha) \arrow[r, "e"] \arrow[d, "a"'] & (B,\beta) \arrow[d, "b"] \arrow[dl, dashed, "w"'] \\
(C,\gamma) \arrow[r, "m"] & (D,\delta)
\end{tikzcd}$$

then there exists a unique morphism $w : (B,\beta) \longrightarrow (C,\gamma)$ in $\mathcal{C}_S$ such that $mw = b$ and $we = a$.

Definition 4.6. We call a monomorphism $m : C \longrightarrow D$ in $\mathcal{C}$ a fibered/cofibered strong monomorphism over an object S if m is strong monomorphism in $\mathcal{C}_S$ / $\mathcal{C}^S$.

Proposition 4.18. If an epimorphism $e : A \longrightarrow B$ is cofibered strong epimorphism over S, then it is strong epimorphism.

Proof. Let

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "a"] & B \arrow[d, "b"] \\ C \arrow[r, "m"] & D \end{tikzcd} \qquad \ldots (4.a)$$

be a commutative diagram with a monomorphism m in $\mathcal{C}$. Now, since e is cofibered strong epimorphism, there exist morphisms $\alpha : S \longrightarrow A$, $\beta : S \longrightarrow A$ such that $S \xrightarrow{\alpha} A \xrightarrow{f} B = S \xrightarrow{\beta} A$.

Now , consider

$$S \xrightarrow{\Upsilon} C = S \xrightarrow{\alpha} A \xrightarrow{a} C \Longrightarrow a \in \mathcal{C}^S$$

$$S \xrightarrow{\delta} D = S \xrightarrow{\beta} B \xrightarrow{b} D \Longrightarrow b \in \mathcal{C}^S$$

and

$$S \xrightarrow{\Upsilon} C \xrightarrow{m} D = S \xrightarrow{\alpha} A \xrightarrow{a} C \xrightarrow{m} D = S \xrightarrow{\alpha} A \xrightarrow{e} B \xrightarrow{b} D$$

$$= S \xrightarrow{\beta} B \xrightarrow{b} D = S \xrightarrow{\delta} D$$

$\Longrightarrow m \in \mathcal{C}^S$ and , by lemma 1.19 , m is a monomorphism in $\mathcal{C}^S$. Thus m we have the following commutative diagram in $\mathcal{C}^S$

$$\begin{tikzcd} (\alpha, A) \arrow[r, "e"] \arrow[d, "a"] & (\beta, B) \arrow[d, "b"] \arrow[dl, dashed, "w"'] \\ (\Upsilon, C) \arrow[r, "m"] & (\delta, D) \end{tikzcd} \qquad \ldots (4.b)$$

Now, since e is cofibered strong epimorphism over S, there exists a unique morphism $w : (\beta, B) \longrightarrow (\gamma, C)$ in $\mathcal{C}^S$ such that $mw = b$ and $we = a$ in $\mathcal{C}^S$ and hence there exists a $w : B \longrightarrow C$ in $\mathcal{C}$ such that $mw = b$, $we = a$ in $\mathcal{C}$ $\Longrightarrow$ e is strong epimorphism.

Similarly , we can prove

Proposition 4.19. If a monomorphism m is fibered strong monomorphism over an object S , then it is strong monomorphism.

Now , we give converse for the above propositions :

Proposition 4.20. Let $\mathcal{C}$ be a V-category and $e : A \longrightarrow B$ be a strong epimorphism. Then f is fibered strong epimorphism over S.

Proof. Since $\mathcal{C}$ is a V-category , there exists a morphism $\beta : B \longrightarrow S$ in $\mathcal{C}$. Then considering $\alpha : A \longrightarrow S = A \xrightarrow{e} B \xrightarrow{\beta} S$ condition (i) of definition 4.5 is satisfied.

Next , let

$$\begin{tikzcd}
(A,\alpha) \arrow[r, "e"] \arrow[d, "a"] & (B,\beta) \arrow[d, "b"] \\
(C,\gamma) \arrow[r, "m"] & (D,\delta)
\end{tikzcd} \qquad \ldots(4.\ddot{o})$$

be a commutative square with a monomorphism m in $\mathcal{C}_S$.
Then , we have the following commutative square in $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "a"] & B \arrow[d, "b"] \\ C \arrow[r, "m"] & D \end{tikzcd} \qquad ...(4.d)$$

Since m is mono in $\mathcal{C}_S$, by lemma 1.18, it is mono in $\mathcal{C}$ and hence, there exists a unique morphism $w : B \longrightarrow C$ such that $we = a$, $mw = b$.

Now,

$$B \xrightarrow{w} C \xrightarrow{\gamma} S = B \xrightarrow{w} C \xrightarrow{m} D \xrightarrow{\delta} S = B \xrightarrow{b} D \xrightarrow{\delta} S = B \xrightarrow{\beta} S$$

$\Longrightarrow w : (B,\beta) \longrightarrow (C,\gamma)$ is unique in $\mathcal{C}_S$ such that $we = a$ and $mw = b$ in $\mathcal{C}_S$.

Dually , we have the following proposition :

Proposition 4.21. Let $\mathcal{C}$ be a V-category and m be a strong monomorphism. Then m is a cofibered strong monomorphism over S.

The following theorems are obvious by proposition 2.1.

Theorem 4.3. An epimorphism e is strong epimorphism if and onlyif f is fibered/cofibered strong epimorphism over a terminal/ initial object.

Theorem 4.4. A monomorphism m is strong monomorphism if and only if m is fibered/cofibered strong monomorphism over a terminal/initial object.

Proposition 4.22. A fibered regular epimorphism over an object S is fibered strong epimorphism over S.

Proof. Let f be fibered regular epimorphism over S ==> f is regular epimorphism in $\mathcal{C}_S$ and since we know that every regular epimorphism in a categories strong epimorphism , implies f is strong epimorphism in $\mathcal{C}_S$. i.e., f is fibered strong epimorphism over S.

Similarly , we can prove the following propositions :

Proposition 4.23. A cofibered regular epimorphism over S is cofibered strong epimorphism over S.

Proposition 4.24. A fibered/cofibered regular monomorphism over an object S is fibered/cofibered strong monomorphism over S.

Now , we investigate certain properties of fibered/cofibered strong epimorphisms and monomorphisms.

Proposition 4.25. All fibered/cofibered strong epimorphisms over any object S are closed under composition and right division.

Proof. Let $A \xrightarrow{e} B \xrightarrow{e'} C$ be two morphisms in $\mathcal{C}$, which are fibered strong epimorphism over an object S ==> there exists morphisms $\beta : B \longrightarrow S$ and $\gamma : C \longrightarrow S$ such that $B \xrightarrow{e'} C \xrightarrow{\gamma} S = B \xrightarrow{\beta} S$. Now , considering $\alpha : A \longrightarrow S = A \xrightarrow{e} B \xrightarrow{\beta} S$, we have

$$(A,\alpha) \xrightarrow{e} (B,\beta) \xrightarrow{e'} (C,\gamma) \in \mathcal{C}_S .$$

Now , since e and e' are fibered strong epimorphism over S , e and e' are strong epimorphisms in $\mathcal{C}_S$. Therefore , as strong epimorphisms are closed under composition ([10] , prop. 3.1), e'e is strong epimorphism in $\mathcal{C}_S$. That is , e'e is fibered strong epimorphism over S.

Other part also follows similarly since strong epimorphisms are closed under right division ([10]) .

Dually, we have the following proposition :

Proposition 4.26. Fibered/cofibered strong monomorphisms are closed under composition and left division.

4.4. Fibered and cofibered canonical factorizations

In this section, we define fibered and cofibered canonical factorizations over an object and observe that in V-categories, if a morphism has canonical epi/mono factorization, then it has fibered epi/cofibered mono canonical factorization over an objects. Also over a terminal/initial object, fibered/cofibered canonical factorizations are the same as canonical factorizations.

Definition 4.7. If a morphism $f : A \longrightarrow B$ in $\mathcal{C}$ has a factorization $f = m\,e$, where e is fibered/cofibered strong epimorphism over S and m is a monomorphism in $\mathcal{C}_S$ / $\mathcal{C}^S$,then we say that f has a fibered/cofibered canonical epifactorization over S.

Definition 4.8. We say that a morphism $f : A \longrightarrow B$ in $\mathcal{C}$ has a fibered/cofibered canonical monofactorization over S if f has a canonical factorization in $\mathcal{C}_S$ / $\mathcal{C}^S$.

Proposition 4.27. Let $\mathcal{C}$ be a V-category and $f : A \longrightarrow B$ in $\mathcal{C}$ has a canonical epifactorization. Then f has fibered canonical epifactorization over any object S of $\mathcal{C}$.

Proof. As $\mathcal{C}$ is a V-category there exists a $\beta : B \longrightarrow S$, then considering $\alpha : A \longrightarrow S = A \xrightarrow{f} B \xrightarrow{\beta} S$, we have $f \in \mathcal{C}_S$.

Next, let $A \xrightarrow{f} B = A \xrightarrow{e} A' \xrightarrow{m} B$ be a canonical factorization of f in $\mathcal{C}$. Considering

$\alpha' : A' \longrightarrow S = A' \xrightarrow{m} B \xrightarrow{\beta} S$, we have a factorization

$$(A,\alpha) \xrightarrow{f} (B,\beta) = (A,\alpha) \xrightarrow{e} (A',\alpha') \xrightarrow{m} (B,\beta)$$

in $\mathcal{C}_S$. By proposition 4.22, e is strong epimorphism in $\mathcal{C}_S$. That is , fibered strong epimorphism over S and , by Lemma 1.18, m is mono in $\mathcal{C}_S$. Thus $A \xrightarrow{e} A' \xrightarrow{m} B$ is a fibered canonical epifactorization over S.

Dually, we have the following proposition :

Proposition 4.28. Let $\mathcal{C}$ be a V-category and a morphism $f : A \longrightarrow B$ in $\mathcal{C}$ has canonical monofactorization. Then f has cofibered canonical monofactorization over any object S in $\mathcal{C}$.

The following theorems are true because of theorems 4.5, 4.6 and Lemmas 1.18 and 1.19.

Theorem 4.7. A morphism $f : A \longrightarrow B$ in $\mathcal{C}$ has a canonical epifactorization iff f has fibered/cofibered canonical epifactorization over a terminal/initial object.

Theorem 4.8. A morphism $f : A \longrightarrow B$ in $\mathcal{C}$ has a canonical monofactorization iff f has fibered/cofibered canonical monofactorization over a terminal/initial object.

4.5. Fibered and cofibered extremal epimorphisms and monomorphisms

In this section, we define fibered and cofibered extremal epimorphisms, monomorphisms over an object of a category and find that (i) in V-categories, an extremal epimorphism/monomorphism is fibered/cofibered extremal epimorphism/monomorphism over all objects of the category, (ii) a fibered/cofibered extremal epimorphism/monomorphism is extremal epimorphism/monomorphism, (iii) over terminal/initial objects, fibered/cofibered extremal epimorphisms and monomorphisms are extremal epimorphisms and monomorphisms, and (iv) A fibered strong epimorphism over S is fibered extremal epimorphism over S.

Definition 4.9. An epimorphism $e : A \longrightarrow B$ in $\mathcal{C}$ is called a fibered/cofibered extremal epimorphism over an object S if e is extremal epimorphism in $\mathcal{C}_S$ / $\mathcal{C}^S$ i.e. (i) there exist

$\alpha : A \longrightarrow S$ and $\beta : B \longrightarrow S$ such that $A \xrightarrow{e} B \xrightarrow{\beta} S = A \xrightarrow{\alpha} S$

and (ii) if

$$(A,\alpha) \xrightarrow{e} (B,\beta) = (A,\alpha) \xrightarrow{e'} (A',\alpha') \xrightarrow{m'} (B,\beta)$$

and m' a monomorphism in $\mathcal{C}_S$ implies m' is an isomorphism in $\mathcal{C}_S$.

Definition 4.10. A monomorphism $m : A \longrightarrow B$ in $\mathcal{C}$ is called a fibered/cofibered extremal monomorphism over an object S of $\mathcal{C}$ if m is extremal monomorphism in $\mathcal{C}_S / \mathcal{C}^S$ i.e. (i) there exist $\alpha : A \longrightarrow S$ and $\beta : B \longrightarrow S$ such that $A \xrightarrow{m} B \xrightarrow{\beta} S = A \xrightarrow{\alpha} S$ and (ii) if

$$(A,\alpha) \xrightarrow{m} (B,\beta) = (A,\alpha) \xrightarrow{e'} (A',\alpha') \xrightarrow{m'} (B,\beta)$$

with e' an epimorphism in $\mathcal{C}_S$ implies e' is an isomorphism.

Proposition 4.29. Let $\mathcal{C}$ be a V-category and $e : A \longrightarrow B$ be an extremal epimorphism. Then e is fibered extremal epimorphism over all objects S of $\mathcal{C}$.

Proof. Since $\mathcal{C}$ is a V-category, there exists $\beta : B \longrightarrow S$ in $\mathcal{C}$, then considering $\alpha : A \longrightarrow S = A \xrightarrow{e} B \xrightarrow{\beta} S$, we see $e \in \mathcal{C}_S$.

Next, if

$$(A,\alpha) \xrightarrow{e} (B,\beta) = (A,\alpha) \xrightarrow{e'} (A',\alpha') \xrightarrow{m'} (B,\beta)$$

with m' a monomorphism in $\mathcal{C}_S$, then, by Lemma 1.18, m' is a monomorphism in $\mathcal{C}$ and $A \xrightarrow{e} B = A \xrightarrow{e'} A' \xrightarrow{m'} B \Longrightarrow m'$ is an isomorphism in $\mathcal{C}$ and hence in $\mathcal{C}_S$ by Lemma 4.3 $\Longrightarrow$ e is fibered extremal epi over S. ||

Dually, we have the following proposition :

Proposition 4.30. Let $\mathcal{C}$ be a V-category and m be an extremal monomorphism . Then m is cofibered extremal monomorphism over all objects S of $\mathcal{C}$.

Proposition 4.31. Every fibered extremal epimorphism over an object S is extremal epimorphism.

Proof. Let e : $A \longrightarrow B$ be a fibered extremal epimorphism over an object S $\Longrightarrow$ there exist $\alpha : A \longrightarrow S$ and $\beta : B \longrightarrow S$ such that $A \xrightarrow{e} B \xrightarrow{\beta} S = A \xrightarrow{\alpha} S$. Next , let

$A \xrightarrow{e} B = A \xrightarrow{e'} A' \xrightarrow{m'} B$ with m' a monomorphism in .

Then, considering $\alpha' : A' \longrightarrow S = A' \longrightarrow B \longrightarrow S$, we have

$(A,\alpha) \xrightarrow{e} (B,\beta) = (A,\alpha) \xrightarrow{e'} (A',\alpha') \xrightarrow{m'} (B,\beta)$ in $\mathcal{C}_S$.

Since m' is monomorphism in $\mathcal{C}$, hence in $\mathcal{C}_S$ ===> m' is an isomorphism in $\mathcal{C}_S$, hence in $\mathcal{C}$ ===> f is extremal epimorphism.

Dually, we have the following :

Proposition 4.32. Every cofibered extremal monomorphism over an object S is extremal monomorphism.

Proposition 4.33. Every cofibered extremal epimorphism over an object S is extremal epimorphism.

Proof. Let $e : A \longrightarrow B$ be a cofibered extremal epimorphism over an object S. Then there exist $\alpha : S \longrightarrow A$ and $\beta : S \longrightarrow B$ such that $S \xrightarrow{\alpha} A \xrightarrow{e} B = S \xrightarrow{\beta} B$. Next, if

$$A \xrightarrow{e} B = A \xrightarrow{e'} A' \xrightarrow{m'} B$$

with m' a monomorphism, then , considering $S \xrightarrow{\alpha'} A' = S \xrightarrow{\alpha} A \xrightarrow{e'} A'$ we have by Lemma 1.19,

$$(\alpha,A) \xrightarrow{e} (\beta,B) = (\alpha,A) \xrightarrow{e'} (\alpha',A') \xrightarrow{m'} (\beta,B)$$

with m' a monomorphism in $\mathcal{C}^S$ ==> [illegible], m' is an isomorphism in $\mathcal{C}^S$ and hence in $\mathcal{C}$ $\Longrightarrow$ $e : A \longrightarrow B$ is extremal epimorphisms.

Dually, we have the following :

Proposition 4.34. Every fibered extremal monomorphism over an object S is extremal monomorphism .

The following theorems are consequences of Proposition 2.1.

Theorem 4.9. An epimorphism e is extremal epimorphism if and only if e is fibered/cofibered extremal epimorphism over an object S.

Theorem 4.10. A monomorphism m is extremal monomorphism if and only if m is fibered/cofibered extremal monomorphism over an object S.

Proposition 4.35. A fibered strong epimorphism over an object S is fibered extremal epimorphism over S.

Proof. Let $e : A \longrightarrow B$ be fibered strong epimorphism over S. Then there exist $\alpha : A \longrightarrow S$ and $\beta : B \longrightarrow S$ such that $A \xrightarrow{e} B \xrightarrow{\beta} S = A \xrightarrow{\alpha} S$.

Next, let

$$(A,\alpha) \xrightarrow{e} (B,\beta) = (A,\alpha) \xrightarrow{e'} (A',\alpha') \xrightarrow{m'} (B,\beta)$$

in $\mathcal{C}_S$ with m' a monomorphism in $\mathcal{C}_S$.

Then considering the following diagram

$$\begin{tikzcd}
(A,\alpha) \arrow[r, "e"] \arrow[d, "e'"'] & (B,\beta) \arrow[d, "Id"] \arrow[dl, dashed, "w"'] \\
(A',\alpha') \arrow[r, "m'"] & (B,\beta)
\end{tikzcd}$$

there exists a unique morphism w : (B,β) ---> (A',α') in $\mathcal{C}_S$ such that m'w = Id ==> m' is a retraction in $\mathcal{C}_S$ and therefore it is an isomorphism by proposition 5.1* ([13] Ch-1).

Similarly, we have the following propositions which can be easily checked.

Proposition 4.36. A cofibered strong epimorphism over S is cofibered extremal epimorphism over S.

Proposition 4.37. A fibered/cofibered strong monomorphism over S is fibered/cofibered extremal monomorphism over S.

4.6. Fibered and cofibered extremal epi-monofacterizations

In this section, we define fibered and cofibered extremal epi-mono facterizations over an object S and observe that in a

V-category , a morphism has extremal epi-mono factorization if and only if the morphism has fibered extremal epi-mono factorization over any object , and dually, similarly for extremal mono-epi factorization. Now we define

Definition 4.11. A morphism $f : A \longrightarrow B$ in $\mathcal{C}$ is said to have a fibered/cofibered extremal epi-mono factorization over an object S if f has extremal epi-mono factorization in $\mathcal{C}_S / \mathcal{C}^S$ i.e. in first case if (i) there exist $\alpha : A \longrightarrow S$ and $\beta : B \longrightarrow S$ such that $A \xrightarrow{f} B \xrightarrow{\beta} S = A \xrightarrow{\alpha} S$ and (ii) there exists a factorization

$$(A,\alpha) \xrightarrow{f} (B,\beta) = (A,\alpha) \xrightarrow{g} (A',\alpha') \xrightarrow{h} (B,\beta)$$

such that g is extremal epi in $\mathcal{C}_S$ and h is mono in $\mathcal{C}_S$.

Definition 4.12. A morphism $f : A \longrightarrow B$ in $\mathcal{C}$ is said to have a fibered/cofibered epi- extremal mono factorization over S if f has epi- extremal mono factorization in $\mathcal{C}_S / \mathcal{C}^S$.

Proposition 4.38. Let $\mathcal{C}$ be a V-category and f has extremal epi-mono factorization. Then f has fibered extremal epi-mono factorization.

Proof. Let $f : A \longrightarrow B$ has extremal epi-mono factorization

$A \xrightarrow{f} B = A \xrightarrow{g} A' \xrightarrow{h} B$. Now as $\mathcal{C}$ is a V-category, there exists $\beta : B \longrightarrow S$, and considering

$\alpha : A \longrightarrow S = A \xrightarrow{f} B \xrightarrow{\beta} S$, we have first condition of the definition. Next, considering $\alpha' = \beta h : A' \longrightarrow S$, we have

$$(A,\alpha) \xrightarrow{f} (B,\beta) = (A,\alpha) \xrightarrow{g} (A',\alpha') \xrightarrow{h} (B,\beta)$$

in $\mathcal{C}_S$. Since g is extremal epi in a V-category $\mathcal{C}$, by Proposition 4.29, g is fibered extremal epi over S and by Lemma 1.18 , h is mono in $\mathcal{C}_S$. Hence $A \xrightarrow{g} A \xrightarrow{h} B$ is fibered extremal epi- mono factorization.

Dually, we have the following :

Proposition 4.39. If $\mathcal{C}$ is a V-category and f has a epi-extremal mono factorization, then f has cofibered epi-extremal mono factorization over S.

We also have :

Proposition 4.40. If f has fibered extremal epi-mono factorization over S in $\mathcal{C}$, then f has extremal epi- mono factorization in $\mathcal{C}$.

Proof. Let

$$(A,\alpha) \xrightarrow{f} (B,\beta) = (A,\alpha) \xrightarrow{g} (A',\alpha') \xrightarrow{h} (B,\beta)$$

be extremal epi- mono factorization of f is $\mathcal{C}_S$. Now, by Proposition 4.31, g is extremal epi in $\mathcal{C}$ and, by Lemma 1.18, h is mono in $\mathcal{C}$. Hence $f : A \longrightarrow B = A \xrightarrow{g} A' \xrightarrow{h} B$ is extremal epi-mono factorization.

Dually, we have :

Proposition 4.41. If f has cofibered epi- extremal mono factorization , then f has epi- extremal mono factorization.

References

Bucur [1] , Herrilich [5] , Isbell [6], [7] , Kelly [10] , Mitchell [13] .

CHAPTER V

SOME GENERALIZATIONS OF CATEGORIES OVER AND BELOW OBJECTS

5.0. Introduction. In this Chapter, we, in the first instant, define categories above and below objects for more than one object. In section 5.1, we introduce these categories for two objects ,whereas in section 5.2 for a countable number of objects. But in section 5.3, we define a category, which is both above and below objects at the same time. In the last section, we construct certain categories over and below categories similar to the comma categories introduced by Stäuffer [15] . In section 5.1, we observe that these categories preserve certain properties like products,coproducts,equalizers, coequalizers, completeness and co-completeness as for those with a single object. Similarly, in section 5.2, we show beside other investigations that category $\mathcal{C}_{S_1 \ldots S_n \ldots} = \mathcal{C}_{\Pi S_i}$, where ΠS_i is the product of a family $\{S_i\}_{i \in I}$ of object of $\mathcal{C}$. Actually the category $\mathcal{C}_{S_1}^{S_2}$ behaves like $\mathcal{C}_{S_1}$ or $\mathcal{C}^{S_2}$, and $\mathcal{C}_{S_1}^{S_2}$ is normal and abelian if S_1 is terminal object and S_2 is an initial object. The categories constructed in the end give the categories introduced in earlier sections as particular cases or subcategories.

5.1. Categories $\mathcal{C}_{S_1,S_2}$ and $\mathcal{C}^{S_1,S_2}$

In this section, we define categories over two objects and below two objects and observe that the properties, which hold for a category for a single object, also hold for these cases using lemmas and propositions for single object, except that in the cases of products/coproducts, normality/conormality and abelianhess, however after certain additional conditions, the situation can be brought under control.

Definition 5.1. Let $\mathcal{C}$ be a category and S_1 and S_2 be two objects of $\mathcal{C}$. Then the category $\mathcal{C}_{S_1,S_2}$ consists of

(i) The class of objects; The triples (A_1,α_1,α_2), A is an object of $\mathcal{C}$ and $\alpha_i : A \longrightarrow S_i$, $i = 1,2$, are morphism in $\mathcal{C}$.

(ii) The class of morphisms : If (A_1,α_1,α_2) and (B,β_1,β_2) are two objects of $\mathcal{C}_{S_1,S_2}$, then the set of morphisms from (A_1,α_1,α_2) to (B,β_1,β_2) consists of all morphisms $\gamma : A \longrightarrow B$ in $\mathcal{C}$ such that the follwing diagrams, for $i = 1,2$

$$\begin{tikzcd}
A \arrow[dr, "\alpha_i"] \arrow[d, "\gamma"'] & \\
B \arrow[r, "\beta_i"'] & S_i
\end{tikzcd}$$

Proof. Define covariant functors

$$F_i : \mathcal{C}_{S_1,S_2} \longrightarrow \mathcal{C}_{S_i} , \; i = 1,2$$

such that

(i) $F_i (A_1, \alpha_1,\alpha_2) = (A , \alpha_i)$

and

(ii) For any morphism $f : (A , \alpha_1,\alpha_2) \longrightarrow (B,\beta_1,\beta_2)$ in $\mathcal{C}_{S_1,S_2}$

$F_i(f) = f : (A, \alpha_i) \longrightarrow (B,\beta_i)$ in $\mathcal{C}_{S_i}$

Obviously, F_i's , being identity functors for morphisms, are embeddings in the sense of Freyd.

Similarly, the other part follows .

Proposition 5.2. If $\mathcal{C}$ has an initial/ a terminal object, then $\mathcal{C}_{S_1,S_2}$ / $\mathcal{C}^{S_1,S_2}$ also has an initial/a terminal object.

Proof. Since $\mathcal{C}$ has an initial object, therefore, by proposition 1.1, $\mathcal{C}_{S_1}$ has an initial object and hence by repearing the argument $\mathcal{C}_{S_1,S_2}$ has initial object. The dual case is also similarly available.

Proposition 5.3. If $\mathcal{C}$ has ceproducts/products, then

$\mathcal{C}_{S_1,S_2}$ / $\mathcal{C}^{S_1,S_2}$ also has coproducts/products.

Proof. If $\mathcal{C}$ has coproducts , then by Lemma 1.1, $\mathcal{C}_{S_1}$ has coproducts and applying the lemma again, we get the result.

Proposition 5.4. If $\mathcal{C}$ has equalizers, then $\mathcal{C}_{S_1,S_2}$ and $\mathcal{C}^{S_1,S_2}$ has equalizers.

Proof. If $\mathcal{C}$ has equalizers, then, by Lemma 1.3, $\mathcal{C}_{S_1}$ has equalizers, again, using the same lemma 1.3, $\mathcal{C}_{S_1,S_2}$ has equalizers.

Similarly, we have the following propositions using the lemma 1.4 .

Proposition 5.5. If $\mathcal{C}$ has coequalizers, then $\mathcal{C}_{S_1,S_2}$ and $\mathcal{C}^{S_1,S_2}$ also have coequalizers.

Proposition 5.6. If $\mathcal{C}$ has pullbacks, pushouts, then $\mathcal{C}_{S_1,S_2}$ and $\mathcal{C}^{S_1,S_2}$ also have pullbacks , pushouts.

(For proof, apply Lemmas 1.5 and 1.6 repeatedly)

Proposition 5.7. If $\mathcal{C}$ has intersections, cointersections, then $\mathcal{C}_{S_1,S_2}$ and $\mathcal{C}^{S_1,S_2}$ also have intersections, cointersections.

(For proof, apply Lemmas 1.7 and 1.8 repeatedly)

Now, the above propositions lead to the following theorem :

Theorem 5.1. If $\mathcal{C}$ is cocomplete/complete category, then so $\mathcal{C}_{S_1,S_2}$ / $\mathcal{C}^{S_1,S_2}$ for any two objects S_1 and S_2.

Remark 5.1. As we have proved in Chapter I, that the converse of the above theorem for a single object is true if we use the fact that the object is either terminal or initial , but in the present case , it is not possible , because all categories with objects as algebraic structures (except possibly the category of sets, the category of top. spaces etc.) have only one terminal or initial object. The converse can be proved if we assume that both objects S_1 and S_2 are terminal or initial, but this assumption will be not generally valid.

Therefore, we cannot have such a normal, exact or abelian category.

5.2. Categories over and below with countable number of objects.

In this section, we generalize the categories of section 5.1 as the category $\mathcal{C}_{S_1,S_2,\ldots,S_n}$ and $\mathcal{C}^{S_1,S_2,\ldots,S_n}$, for a countable number of objects $S_1,S_2,\ldots$ of $\mathcal{C}$, and observe that if $\mathcal{C}$ is a category with product then $\mathcal{C}_{S_1,\ldots,S_n,\ldots}$

is equivalent to the category $\mathcal{C}_{\Pi S_i}$. Furthermore, the propositions, which hold for $\mathcal{C}_{S_1,S_2}$, also hold here.(The proof can be obtained by using repeated arguments (induction)).

Definition 5.3. Let $\mathcal{C}$ be a category and $\{S_i\}_{i\in I}$ be a family of countable number of objects of $\mathcal{C}$. Then category $\mathcal{C}_{S_1,\dots,S_n,\dots}$ consists of

(i) The class of objects; All tuples of the form $(A\ ,\ \alpha_1,\alpha_2,\dots)$, A is an object of $\mathcal{C}$ and $\alpha_i : A \longrightarrow S_i$, $\forall i \in I$ are morphisms of $\mathcal{C}$.

(ii) If $(A\ ,\ \alpha_1,\alpha_2,\dots)$ and $(B,\beta_1,\beta_2,\dots)$ are two objects of $\mathcal{C}_{S_1,\dots,S_n,\dots}$ then the set of morphisms from $(A\ ,\ \alpha_1,\alpha_2,\dots)$ to $(B,\ \beta_1,\beta_2,\dots)$ consists of all morphisms $\gamma : A \longrightarrow B$ in $\mathcal{C}$ such that the following diagrams

$$\begin{tikzcd} A \arrow[dr, "\alpha_i"] \arrow[dd, "\gamma"'] & \\ & S_i \\ B \arrow[ur, "\beta_i"'] & \end{tikzcd}$$

are commutative for all i.

Dually, we define $\mathcal{C}^{S_1,\ldots,S_n,\ldots}$ as follows :

Definition 5.4. The category $\mathcal{C}^{S_1,\ldots,S_n,\ldots}$ consists of

(i) The class of objects: All tuples of the form $(\ldots\ \alpha_2,\alpha_1,A)$, A is an object of $\mathcal{C}$ and $\alpha_i : S_i \longrightarrow A$ are morphisms of $\mathcal{C}$ for all i.

(ii) If $(\ \ldots\ \alpha_2,\alpha_1,A)$ and $(\ \ldots\ \beta_2,\beta_1,B)$ are two objects of $\mathcal{C}^{S_1\ldots S_n\ldots}$ then the set of morphisms from $(\ldots,\alpha_2,\alpha_1,A)$ to $(\ldots,\beta_2,\beta_1,B)$ consists of all morphisms $\gamma : A \longrightarrow B$ in $\mathcal{C}$ such that the following diagrams

$$\begin{tikzcd} & A \arrow[dd, "\gamma"] \\ S_i \arrow[ur, "\alpha_i"] \arrow[dr, "\beta_i"'] & \\ & B \end{tikzcd}$$

are commutative for all i.

The following proposition can be proved, proceeding on the lines of proposition 5.1.

Proposition 5.8. The category $\mathcal{C}_{S_1,\ldots,S_n,\ldots} / \mathcal{C}^{S_1,\ldots S_n,\ldots}$ can be embedded in the category $\mathcal{C}_{S_i} / \mathcal{C}^{S_i}$ for any $i \in I$.

An important equivalence theorem is as follows :

Theorem 5.2. Let $\mathcal{C}$ be a category with products and $S = \prod_{i \in I} S_i$ be the product of a countable family of objects $\{S_i\}_{i \in I}$. Then $\mathcal{C}_{S_1,\dots,S_n,\dots} \approx \mathcal{C}_{\prod S_i} = \mathcal{C}_S$.

Proof. Let $(A, \alpha_1, \alpha_2, \dots)$ be an object in $\mathcal{C}_{S_1,\dots,S_n,\dots}$. As S is the product of S_i's there exists a unique morphism $\alpha : A \longrightarrow S$ such that

$$A \xrightarrow{\alpha} S \xrightarrow{p_i} S_i = A \xrightarrow{\alpha_i} S_i \ , \forall i \ ,$$

where $p_i : S \longrightarrow S_i$, $i \in I$, are canonical projection.

Next, let $f : (A, \alpha_1, \alpha_2, \dots) \longrightarrow (B, \beta_1, \dots)$ be a morphism in $\mathcal{C}_{S_1,S_2,\dots}$. As $\beta_i : B \longrightarrow S_i$, and $S = \prod S_i$, there exist unique $\beta : B \longrightarrow S$ such that

$$B \xrightarrow{\beta} S \xrightarrow{p_i} S_i = B \xrightarrow{\beta_i} S_i \ , \forall i \ .$$

Therefore, we have the following commutative diagram for all i.

$$\begin{tikzcd}
A \arrow[dd, "f"'] \arrow[dr, dashed, "\alpha"'] \arrow[drr, "\alpha_i"] & & \\
& S \arrow[r, "p_i"] & S_i \\
B \arrow[ur, dashed, "\beta"] \arrow[urr, "\beta_i"'] & &
\end{tikzcd}$$

Now, $p_i\beta f = \beta_i f = \alpha_i = p_i\,\alpha$, $\forall i$.

$\Longrightarrow$ $\beta f = \alpha$, since p_i's are unique canonical projections

$\Longrightarrow$ $f : (A,\alpha) \longrightarrow (B,\beta) \in \mathcal{C}_{S_i}$.

Now, we define covariant functors F and F' as

(i) $F : \mathcal{C}_{S_1,\ldots,S_n,\ldots} \longrightarrow \mathcal{C}_S$

such that

(a) $F(A,\alpha_1,\alpha_2,\ldots) = (A,\alpha)$, where $p_i\alpha = \alpha_i$, $\forall i$

(b) and $F(f) = f$, $\forall f : (A,\alpha_1,\alpha_2,\ldots) \longrightarrow (B,\beta_1,\beta_2,\ldots)$ in $\mathcal{C}_{S_1,\ldots,S_n,\ldots}$

Also (ii) $F' : \mathcal{C}_S \longrightarrow \mathcal{C}_{S_1,\ldots,S_n,\ldots}$

such that

(a') $F'(A,\alpha) = (A,p_1\alpha,p_2\alpha,\ldots)$, where $p_i : S \longrightarrow S_i$ are canonical projections

(b') $F'(f) = f : (A,p_1\alpha,\ldots) \longrightarrow (B,p_1\beta\ \ldots)$, $\forall f : (A,\alpha) \longrightarrow (B,\beta)$

Hence,

$$\begin{aligned} F\,F'\,(A,\alpha) &= F(A,p_1\alpha,\ldots) \\ &= (A,\alpha) \end{aligned}$$

Also,

$$F\,F'\,(f) = F(f) = f$$

$$\Longrightarrow F\,F' = I_{\mathcal{C}_S}$$

Similarly,

$$F'F(A,\alpha_1,\ldots,\alpha_n) = F'(A,\alpha),\text{ such that } p_i\alpha = \alpha_i$$

$$= (A, p_1\alpha, p_2\alpha, \ldots)$$

$$= (A, \alpha_1, \alpha_2, \ldots)$$

also,

$$F'\,F(f) = F'(f) = f$$

$$\Longrightarrow F'F = I_{\mathcal{C}_{S_1,\ldots,S_n,\ldots}}$$

Next, the following diagram is commutative

$$\begin{tikzcd}
F'F(A,\alpha_1,\ldots,\alpha_n,\ldots) \arrow[r, "F'F(A,\alpha_1,\ldots,\alpha_n,\ldots)"] \arrow[d, "F'F'(f)"] & (A,\alpha_1,\ldots,\alpha_n,\ldots) \arrow[d, "f"] \\
F'F(B,\beta_1,\ldots,\beta_n,\ldots) \arrow[r, "F'F'(B,\beta_1,\ldots,\beta_n,\ldots)"] & (B,\beta_1,\ldots,\beta_n,\ldots)
\end{tikzcd}$$

$\Longrightarrow$ F is natural equivalence between $\mathcal{C}_S$ and $\mathcal{C}_{S_1,\ldots,S_n,\ldots}$

$\Longrightarrow$ $\mathcal{C}_S \approx \mathcal{C}_{S_1,S_2,\ldots}$, where $S = \prod_{i\in I} S_i$.

Dually, we have the following theorem :

Theorem 5.3. Let $\mathcal{C}$ be a category with coproducts. Then $\mathcal{C}^{S_1,\ldots,S_n,\ldots}$ is equivalent to the category $\mathcal{C}^S$, where $S = \bigoplus S_i$.

The following propositions also hold for these categories (Proof can be obtained by repeated applications of propositions proved in section 5.1) :

Proposition 5.9. If $\mathcal{C}$ has initial/terminal object then $\mathcal{C}_{S_1,\ldots,S_n,\ldots}$/ $\mathcal{C}^{S_1,\ldots,S_n,\ldots}$ also has initial/terminal object.

Proposition 5.10. If $\mathcal{C}$ has coproducts,equalizers,coequalizers, pullbacks, pushouts, intersections and cointersections, then $\mathcal{C}_{S_1,S_2,\ldots,S_n,\ldots}$ also has respective properties.

Remark 5.2. The definitions and results in Chapters III and IV can be modified for more than one object, and multifibered and multicofibered objects and morphisms and other notions can similarly be studied , for example :

Definition 5.5. An object P of a category $\mathcal{C}$ is called multifibered projective over a countable number of objects

$\{S_i\}_{i\in I}$ if P is projective in $\mathcal{C}_{S_1,S_2,\ldots,S_n,\ldots}$ i.e. if (i) there exists a family $\{p_i : P \longrightarrow S_i\}_{i\in I}$ of morphisms in $\mathcal{C}$ and (ii) for any epimorphism $e : (A,\alpha_1,\alpha_2,\ldots) \longrightarrow (B,\beta_1,\beta_2,\ldots)$ and for any morphism $f : (P,p_1,\ldots) \longrightarrow (B,\beta_1,\ldots)$, there exists a unique morphism $\gamma : (P,p_1,\ldots) \longrightarrow (A,\alpha_1,\ldots)$ such that $e\gamma = f$.

5.3. Category over-below objects : $\mathcal{C}_{S_1}^{S_2}$

In this section, we define a category $\mathcal{C}_{S_1}^{S_2}$, where S_1 and S_2 are two objects of a category $\mathcal{C}$, we show '' how $\mathcal{C}_{S_1}^{S_2}$ is related to $\mathcal{C}$ ''. That is , what properties of $\mathcal{C}$ are preserved by $\mathcal{C}_{S_1}^{S_2}$ in a natural way ?

Definition 5.6. Let $\mathcal{C}$ be a category and S_1 and S_2 be two objects of $\mathcal{C}$. Then category $\mathcal{C}_{S_1}^{S_2}$ consists of

(i) The class of objects: All triples like $(\underline{\alpha},A,\bar{\alpha})$,where A is an object of $\mathcal{C}$, $\underline{\alpha} : A \longrightarrow S_1$, and $\bar{\alpha} : S_2 \longrightarrow A$ are morphisms in $\mathcal{C}$.

(ii) If $(\underline{\alpha}_1, A_1, \bar{\alpha}_1)$ and $(\underline{\beta}_1,B_1,\bar{\beta}_1)$ are two objects in $\mathcal{C}_{S_1}^{S_2}$, then the set of morphisms from $(\underline{\alpha}_1, A_1, \bar{\alpha}_1)$ to $(\underline{\beta}_1, B_1, \bar{\beta}_1)$ consists tof all morphisms $\gamma : A_1 \longrightarrow B_1$ in $\mathcal{C}$ such that the following diagram

$$\begin{tikzcd}
 & A_1 \arrow[dd, "\gamma"] \arrow[dr, "\bar{\alpha}_1"] & \\
S_2 \arrow[ur, "\underline{\alpha}_1"] \arrow[dr, "\underline{\beta}_1"'] & & S_1 \\
 & B_1 \arrow[ur, "\bar{\beta}_1"'] &
\end{tikzcd}$$

is commutative.

Proposition 5.11. The category $\mathcal{C}^{S_2}_{S_1}$ can be embedded in both $\mathcal{C}_{S_1}$ and $\mathcal{C}^{S2}$.

Proof. Define covariant functors

$$F : \mathcal{C}^{S_2}_{S_1} \longrightarrow \mathcal{C}_{S_1}, \text{ and } F' : \mathcal{C}^{S_2}_{S_1} \longrightarrow \mathcal{C}^{S_2}$$

such that

$$F(\underline{\alpha}, A, \bar{\alpha}) = (A, \bar{\alpha}) \quad , \quad F'(\underline{\alpha}, A, \bar{\alpha}) = (\underline{\alpha}, A)$$

and

$$F(f) = f \quad , \quad F'(f) = f ,$$

for all morphisms $f : (\underline{\alpha}, A, \bar{\alpha}) \longrightarrow (\underline{\beta}, B, \bar{\beta})$.

Obviously, as in propositions 5.1 and 5.8, F and F' are embeddings in the sense of Freyd [3] .

Proposition 5.12. If $\mathcal{C}$ has coproducts, then $\mathcal{C}_{S_1}^{S_2}$ has coproducts provided S_2 is an initial object , and if $\mathcal{C}$ has products , then $\mathcal{C}_{S_1}^{S_2}$ has products provided S_1 is a terminal object.

Proof. If $\mathcal{C}$ has coproduct then , by Lemma 1.1 $\mathcal{C}_{S_1}$ has coproducts and, since S_2 is initial, then, by Lemma 1.2 , $\mathcal{C}_{S_1}^{S_2}$ has coproducts.

Similarly, the other part can be proved.

Proposition 5.13. If $\mathcal{C}$ has equalizers, coequalizers ,then $\mathcal{C}_{S_1}^{S_2}$ also has equalizers, coequalizers respectively.

Proof. If $\mathcal{C}$ has equalizers, by Lemma 1.5, $\mathcal{C}_{S_1}$ has equalizers and now , since $\mathcal{C}_{S_1}$ has equalizers, then by Lemma 1.8, $\mathcal{C}_{S_1}^{S_2}$ has equalizers. Other part can be proved similarly.

We can also obtain the following propositions using previous results :

Proposition 5.14. If $\mathcal{C}$ has pullbacks, pushouts, then $\mathcal{C}_{S_1}^{S_2}$ also has pullbacks, pushouts respectively.

Proposition 5.15. If $\mathcal{C}$ has intersections, cointersections, then $\mathcal{C}_{S_1}^{S_2}$ also has intersections, cointersections respectively.

Now, we have the following theorems obviously available from above propositions 5.12, 5.13 and theorems 1.1, 1.3, and 1.4.

Theorem 5.4. (i) If $\mathcal{C}$ is right complete category, then $\mathcal{C}_{S_1}^{S_2}$ is right complete if S_1 is terminal object of $\mathcal{C}$.

(ii) If $\mathcal{C}$ is a left complete category, then $\mathcal{C}_{S_1}^{S_2}$ is left complete category if S_2 is an initial object.

Theorem 5.5. (i) If $\mathcal{C}$ is normal, then $\mathcal{C}_{S_1}^{S_2}$ is normal if S_1 is terminal and S_2 is initial object.

(ii) If $\mathcal{C}$ is ~~nenormal~~ , then $\mathcal{C}_{S_1}^{S_2}$ is conormal if S_1 is terminal and S_2 is initial object.

These two theorems imply :

Theorem 5.6. If $\mathcal{C}$ is an abelian category, then $\mathcal{C}^{S_2}_{S_1}$ is abelian category if S_1 is terminal and S_2 is an initial object.

5.4. Some more constructions on categories and functors.

Lastly, in this section, we construct some special types of categories based on the following notion introduced by Stauffer [14] as '' Comma category '' . We have constructed

$${}^{\mathcal{A}}_{\mathcal{B}}S\ ,\quad S^{\mathcal{A}}_{\mathcal{B}}\ ,\ {}^{\mathcal{A}}_{\mathcal{B}}\mathcal{C}\ ,\ \mathcal{C}^{\mathcal{A}}_{\mathcal{B}}\ ,\quad {}^{\mathcal{A}}\mathcal{B}_S\ ,\quad S^{\mathcal{B}^{\mathcal{A}}}\ ,\quad {}^{\mathcal{A}}\mathcal{B}_{\mathcal{C}} \text{ and } \mathcal{C}^{\mathcal{B}^{\mathcal{A}}}.$$

Construction 5.1. Let $T : \mathcal{C} \longrightarrow \mathcal{D}$ be a covariant functor and D be an object of $\mathcal{D}$, then a <u>comma category</u> (T,D) consists of

(i) The class of objects as all pairs of the form (A,α), where A is an object of $\mathcal{C}$ and $\alpha : T(A) \longrightarrow D$ is a morphism in $\mathcal{D}$.

(ii) The class of all morphisms , where the set of morphisms for the pair of objects (A,α) to (A',α') is all morphisms $\Upsilon : A \longrightarrow A'$ in $\mathcal{C}$ such that the following diagram

$$\begin{tikzcd}
T(A) \arrow[dr, "\alpha"] \arrow[dd, "T(\Upsilon)"'] & \\
& D \\
T'(A') \arrow[ur, "\alpha'"'] &
\end{tikzcd}$$

is commutative.

The composition of morphisms is defined in the usual way.

Dually , we have the co-comma category (D,T).

Now, we give below some generalizations of the above categories :

Construction 5.2. Let F and G as in $\mathcal{A} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B}$ be covariant functors and S be an object of $\mathcal{C}$. Then we construct a category, which we denote $\frac{\mathcal{A}}{\mathcal{B}}S$ as follows:

(i) The class of objects of $\frac{\mathcal{A}}{\mathcal{B}}S$ consists of all 4-tuples $(A_1,\alpha_1,B_1,\beta_1)$, where A_1 is an object of $\mathcal{A}$, B_1 is an object of $\mathcal{B}$, $\alpha_1 : F(A_1) \longrightarrow S$ and $\beta_1 : G(B_1) \longrightarrow S$ are morphisms in $\mathcal{C}$.

(ii) The set of morphisms from an object $(A_1,\alpha_1,B_1,\beta_1)$ to an object $(A_2,\alpha_2,B_2,\beta_2)$ is a set of all morphisms pairs $(f,g) \in \mathcal{A} \times \mathcal{B}$ such that the following diagram is commutative :

$$\begin{tikzcd}
F(A_1) \arrow[dr, "\alpha_1"] \arrow[dd, "F(f)"'] & & G(B_1) \arrow[dl, "\beta_1"'] \arrow[dd, "G(g)"] \\
& S & \\
F(A_2) \arrow[ur, "\alpha_2"'] & & G(B_2) \arrow[ul, "\beta_2"]
\end{tikzcd}$$

The composition of morphism is componentwise composition.

Dually, we have the following category $S^{\mathcal{A}}_{\mathcal{B}}$:

Construction 5.3. The category $S^{\mathcal{A}}_{\mathcal{B}}$ consists of

(i) The class of objects consists of all tuples of the from $(\alpha_1, A_1, \beta_1, B_1)$, where A_1 and B_1 are objects of $\mathcal{A}$ and $\mathcal{B}$ respectively and $\alpha_1 : S \longrightarrow F(A_1)$ and $\beta_1 : S \longrightarrow G(B_1)$ are morphisms in $\mathcal{C}$.

(ii) The set of morphisms from $(\alpha_1, A_1, \beta_1, B_1)$ to $(\alpha_2, A_2, \beta_2, B_2)$ consists of all morphisms pairs (f, g) in $\mathcal{A} \times \mathcal{B}$ such that the following diagram

$$\begin{tikzcd}
F(A_1) \arrow[dd, "F(f)"] & & G(B_1) \arrow[dd, "G(g)"] \\
& S \arrow[ul, "\alpha_1"'] \arrow[dl, "\alpha_2"] \arrow[ur, "\beta_1"] \arrow[dr, "\beta_2"'] & \\
F(A_2) & & G(B_2)
\end{tikzcd}$$

is commutative.

Remark 5.3. Maclane in his book '' Categories for working mathematicians'' has introduced a similar type of category and denoted it by ($F \downarrow G$), and dually ($F \uparrow G$), He obtains the followings as particular cases of these category

(i) Comma category

(ii) $\mathcal{C}_S$ for a single object

(iii) category of morphisms ($\mathcal{C} \downarrow \mathcal{C}$).

Now, we construct a category over and below a category and denote it as ${}^{\mathcal{A}}_{\mathcal{B}}\mathcal{C}$.

Construction 5.4. Let $F : \mathcal{A} \longrightarrow \mathcal{C}$, $G : \mathcal{B} \longrightarrow \mathcal{C}$ be two covariant functors. Then the category ${}^{\mathcal{A}}_{\mathcal{B}}\mathcal{C}$ consists of

(i) The class of objects of ${}^{\mathcal{A}}_{\mathcal{B}}\mathcal{C}$ consists of all triples $(A_1, \alpha_1, S_1, B_1, \beta_1)$, where $A_1 \in \mathcal{A}$, $S_1 \in \mathcal{C}$, $B_1 \in \mathcal{B}$, and $\alpha_1 : F(A_1) \longrightarrow S_1$, $\beta_1 : G(B_1) \longrightarrow S_1$ are morphisms in $\mathcal{C}$.

(ii) The set of morphisms from anoobject $(A_1, \alpha_1, S_1, B_1, \beta_1)$ to an object $(A_2, \alpha_2, S_2, B_2, \beta_2)$ consists of all triples of morphisms (f, s, g) $\in \mathcal{A} \times \mathcal{C} \times \mathcal{B}$ such that the following diagram

$$\begin{tikzcd}
F(A_1) \arrow[r, "\alpha_1"] \arrow[d, "F(f)"] & S_1 \arrow[d, "s"] & G(B) \arrow[l, "\beta_1"'] \arrow[d, "G(g)"] \\
F(A_2) \arrow[r, "\alpha_2"] & S_2 & G(B_2) \arrow[l, "\beta_2"']
\end{tikzcd}$$

is commutative.

The composition of morphism has usual componentwise multiplication.

Dually, we have the category $\mathcal{C}_{\mathcal{B}}^{\mathcal{A}}$.

Construction 5.5. The category $\mathcal{C}_{\mathcal{B}}^{\mathcal{A}}$ consists of

(i) The class of objects of $\mathcal{C}_{\mathcal{B}}^{\mathcal{A}}$ consists of all tuples of the form $(\alpha_1, A_1, S_1, \beta_1, B_1)$ such that $A_1 \in \mathcal{A}$, $S_1 \in \mathcal{C}$, $B_1 \in \mathcal{B}$ and $\alpha_1 : S_1 \longrightarrow F(A_1)$, $\beta_1 : S_1 \longrightarrow G(B_1) \in \mathcal{C}$.

(ii) The set of morphisms from an object $(\alpha_1, A_1, S_1, \beta_1, B_1)$ to an object $(\alpha_2, A_2, S_2, \beta_2, B_2)$ are all triples $(f, s, g) \in \mathcal{A} \times \mathcal{C} \times \mathcal{B}$ such that the following diagram is commutative

$$\begin{tikzcd}
F(A_1) \arrow[d, "F(f)"] & S_1 \arrow[l, "\alpha_1"'] \arrow[r, "\beta_1"] \arrow[d, "s"] & G(B_1) \arrow[d, "G(g)"] \\
F(A_2) & S_2 \arrow[l, "\alpha_2"'] \arrow[r, "\beta_2"] & G(B_2)
\end{tikzcd}$$

The morphisms have usual componentwise multiplication law of a composition.

Remark 5.4. All previous constructions $\mathcal{C}_S$, $\mathcal{C}^S$, $\mathcal{C}_{S_1,S_2}$, $\mathcal{C}^{S_1,S_2}$, ${}^{\mathcal{A}}\mathcal{B}^S$, $S^{\mathcal{A}}_{\mathcal{B}}$ are particular cases of the above categories.

Construction 5.6. Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be covariant functors and S be an object of $\mathcal{C}$. Then we have a category ${}^{\mathcal{A}}\mathcal{B}_S$ consists of

(i) The class of objects consists of all triples $(A_1,\alpha_1,B_1,\beta_1)$, where $A_1 \in \mathcal{A}$, $B_1 \in \mathcal{B}$ and $\alpha_1 : F(A_1) \longrightarrow B_1 \in \mathcal{B}$, $\beta_1 : G(B_1) \longrightarrow S \in \mathcal{C}$.

(ii) The set of morphisms for an object $(A_1,\alpha_1,B_1,\beta_1)$ to $(A_2,\alpha_2,B_2,\beta_2)$ comprise all triples $(f,g) \in \mathcal{A} \times \mathcal{B}$ of morphisms such that the following diagram

$$\begin{tikzcd}
GF(A_1) \arrow[r, "G(\alpha_1)"] \arrow[d, "GF(f)"] & G(B_1) \arrow[d, "G(g)"] \arrow[dr, "\beta_1"] & \\
 & & S \\
GF(A_2) \arrow[r, "G(\alpha_2)"] & G(B_2) \arrow[ur, "\beta_2"] &
\end{tikzcd}$$

is commutative.

Dually, we define the category $S^{\mathcal{B}^{\mathcal{A}}}$ as follows :

Construction 5.7. The category $S^{\mathcal{B}^{\mathcal{A}}}$ consists of

(i) The class of objects consists of all tuples $(\alpha_1, A_1, \beta_1, B_1)$, where $A_1 \in \mathcal{A}$, $B_1 \in \mathcal{B}$, $\alpha_1 : B_1 \longrightarrow F(A_1) \in \mathcal{B}$ and $\beta_1 : S \longrightarrow G(B_1) \in \mathcal{C}$

(ii) The set of morphisms consists of all pairs $(f,g) \in \mathcal{A} \times \mathcal{B}$ of morphisms such that the following diagram

$$\begin{tikzcd}
 & G(B_1) \arrow[r, "G(\alpha_1)"] \arrow[d, "G(g)"] & GF(A_1) \arrow[d, "GF(f)"] \\
S \arrow[ur, "\beta_1"] \arrow[r, "\beta_2"'] & G(B_2) \arrow[r, "G(\alpha_2)"] & GF(A_2)
\end{tikzcd}$$

is commutative.

Also, we have the following categories generalizations of the above categories :

Construction 5.8. The category ${}^{\mathcal{A}}\mathcal{B}_{\mathcal{C}}$ consists of

(i) The class of objects of all tuples $(A_1, \alpha_1, B_1, \beta_1, S_1)$, where $A_1 \in \mathcal{A}$, $B_1 \in \mathcal{B}$, $S_1 \in \mathcal{C}$ and $\alpha_1 : F(A_1) \longrightarrow B_1 \in \mathcal{B}$, $\beta_1 : G(B_1) \longrightarrow S_1 \in \mathcal{C}$.

(ii) The set of morphisms from an object $(A_1, \alpha_1, B_1, \beta_1, S_1)$ to $(A_2, \alpha_2, B_2, \beta_2, S_2)$ consists of all triples $(f, g, s) \in \mathcal{A} \times \mathcal{B} \times \mathcal{C}$ of morphisms such that the following diagram

$$\begin{tikzcd} FG(A_1) \arrow[r, "G(\alpha_1)"] \arrow[d, "FG(f)"] & G(B_1) \arrow[r, "\beta_1"] \arrow[d, "G(g)"] & S_1 \arrow[d, "s"] \\ FG(A_2) \arrow[r, "G(\alpha_2)"] & G(B_2) \arrow[r, "\beta_2"] & S_2 \end{tikzcd}$$

is commutative.

Construction 5.9. The category $\mathcal{C}\mathcal{B}^{\mathcal{A}}$ consists of

(i) The class of objects all tuples like $(S_1, \beta_1, B_1, \alpha_1, A_1)$, $S_1 \in \mathcal{C}$, $B_1 \in \mathcal{B}$, $A_1 \in \mathcal{A}$ and $\beta_1 : S_1 \longrightarrow G(B_1) \in \mathcal{C}$, $\alpha_1 : B_1 \longrightarrow F(A_1) \in \mathcal{C}$.

(ii) The set of morphisms from an object $(S_1, \beta_1, B_1, \alpha_1, A_1)$ to an object $(S_2, \beta_2, B_2, \alpha_2, A_2)$ consists of all triples $(s, g, f) \in \mathcal{C} \times \mathcal{B} \times \mathcal{A}$ such that the following diagram is commutative

$$\begin{array}{ccccc}
S_1 & \xrightarrow{\beta_1} & G(B_1) & \xrightarrow{G(\alpha_1)} & GF(A_1) \\
\Big\downarrow{\scriptstyle s} & & \Big\downarrow{\scriptstyle G(g)} & & \Big\downarrow{\scriptstyle GF(f)} \\
S_2 & \xrightarrow{\beta_2} & G(B_2) & \xrightarrow{G(\alpha_2)} & GF(A_2)
\end{array}$$

Comma/cocomma category is a particular case of the above category.

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APPENDIX

A NOTE ON CHARACTERISTIC SUBOBJECTS

By

Virendra Prasad

1.

Huq, S.A. in 1968 in [2] has introduced the notion of fully characteristic subobject of an object A in a locally small category $\mathcal{C}$ with zero object images, products and coproducts as a subobject (B,μ) if there exists a morphism $\theta' \in \mathrm{Hom}(B,B)$ for any $\theta \in \mathrm{Hom}(A,A)$ such that the diagram

$$\begin{tikzcd} B \arrow[r, "\theta'"] \arrow[d, "\mu"] & B \arrow[d, "\mu"] \\ A \arrow[r, "\theta"] & A \end{tikzcd}$$

is commutative. He calls the subobject (B,μ) the characteristic subobject of A if above considerations hold only for invertible morphism $\theta \in \mathrm{Hom}(A,A)$. The purpose of this note is to generalise certain properties of characteristic subgroups to characteristic subobjects by categorical techniques, thus obtaining similar situations is different algebraic structures.

The following proposition follows from definition itself :

<u>Proposition 1.</u> Every fully characteristic subobject of an

object is characteristic subobject.

<u>Proposition 2</u>. Let (B,μ) be a characteristic subobject of an object A of $\mathcal{C}$. Then $(B,\mu) \cong (\theta\,\mu\,B,\, u)$ where $(\theta\,\mu\,B, u)$ is the image of B under $\theta\mu$, and $\theta \in \mathrm{Hom}\,(A,A)$ is an invertible morphism.

<u>Proof</u> : Since $\theta \in \mathrm{Hom}\,(A,A) \Longrightarrow$ there exists a unique morphism $\theta' : B \longrightarrow B$

such that the following diagram is commutative

$$\begin{tikzcd} B \arrow[r, "\theta'"] \arrow[d, "\mu"] & B \arrow[d, "\mu"] \\ A \arrow[r] & A \end{tikzcd}$$

$\Longrightarrow$ we have a $B \xrightarrow{\theta\mu} A = B \xrightarrow{\theta'} B \xrightarrow{\mu} A$ with μ a monomorphism. Now since $\theta\mu(B) \xrightarrow{\mu} A = \mathrm{Im}\,(\theta\mu) \Longrightarrow$ there exist a unique morphism $\alpha : \theta\,\mu(B) \longrightarrow B$ such that the following diagram is commutative

$$\begin{tikzcd} & \theta\mu(B) \arrow[dr, "u"] \arrow[dd, "\alpha"] & \\ B \arrow[ur, "p"] \arrow[dr, "\theta'"'] & & A \\ & B \arrow[ur, "\mu"'] & \end{tikzcd}$$

$\Longrightarrow$ ($\theta\mu(B)$, u) $\subseteq$ (B,μ).

Next since θ is invertible there exist $\phi : A \longrightarrow A$ such that $\theta\phi = I_A$, $\phi\theta = I_A \Longrightarrow$ that ϕ is invertible, hence there exists a morphism $\phi' : B \longrightarrow B$ such that the following diagram is commutative

$$\begin{array}{ccc} B & \xleftarrow{\phi'} & B \\ \mu \downarrow & & \downarrow \mu \\ A & \xleftarrow{\phi} & A \end{array}$$

i.e. $\mu\phi' = \phi\mu$.

Consider $\alpha' = p\phi : B \longrightarrow \theta\,\mu\,B$

then $u\,\alpha' = up\phi' = \theta\mu\phi' = \theta\phi\mu = I_A\,\mu = \mu$.

$\Longrightarrow$ $(B,\mu) \subseteq (\theta\,\mu\,(B), u)$.

Hence $(B,\mu) = (\theta\,\mu(B), u)$.

Proposition 3. If (C , μ) is fully characteristic subobject of B and (B, η) is characteristic subobject of A then ((fully))
(C , $\eta\mu$) is fully characteristic subobject of A.

Proof. Let $\theta : A \longrightarrow A$. Since (B, η) is fully characteristic subobject of A there exists a morphism $\theta' : B \longrightarrow B$ such that the following diagram is commutative

$$\begin{tikzcd} B \arrow[r, "\theta'"] \arrow[d, "\eta"] & B \arrow[d, "\eta"] \\ A \arrow[r, "\theta"] & A \end{tikzcd}$$

Now since (C , μ) is fully characteristic subobject of B there exists a morphism $\theta'' : C \longrightarrow C$ such that the following diagram is commutative

$$\begin{tikzcd} C \arrow[r, "\theta''"] \arrow[d, "\mu"] & C \arrow[d, "\mu"] \\ B \arrow[r, "\theta'"] & B \end{tikzcd}$$

and hence the following diagram is commutative

$$\begin{tikzcd} C \arrow[r, "\theta''"] \arrow[d, "\eta\mu"'] & C \arrow[d, "\eta\mu"] \\ A \arrow[r, "\theta"] & A \end{tikzcd}$$

__Proposition 4.__ Let $\{(B_i,\mu_i)\}_{i\in I}$ be a family of fully characteristic subobject of an object A and $(B', \mu) = \bigcap_{i\in I} (B_i,\mu_i)$.

Then (B',μ) is also fully characteristic subobject of A.

<u>Proof.</u> Let $\theta : A \longrightarrow A$ be a morphism and we have the following diagram

$$\begin{tikzcd}
 & B' \arrow[rr, "\theta'"] \arrow[dl, "v_i"'] \arrow[dd, "\mu"] & & B' \arrow[dl, "v_i"'] \arrow[dd, "\mu"] \\
B_i \arrow[rr, "\theta'_i"] \arrow[dr, "\mu_i"'] & & B_i \arrow[dr, "\mu_i"] & \\
 & A \arrow[rr, "\theta"] & & A
\end{tikzcd}$$

as (B_i,μ_i) is fully characteristic subobject of A, $\forall i \Longrightarrow$ there exists morphism $\theta'_i : B_i \longrightarrow B_i$ such that $\mu_i\theta'_i = \theta\,\mu_i$. Since

$$B' \xrightarrow{\mu} A \xrightarrow{\theta} A = B' \xrightarrow{v_i} B_i \xrightarrow{\mu_i} A \xrightarrow{\theta} A$$

$$= B' \xrightarrow{v_i} B_i \xrightarrow{\theta'_i} B_i \xrightarrow{\mu_i}$$

$$= B' \xrightarrow{\theta'_i v_i} B_i \xrightarrow{\mu_i} A \;,$$

then, by definition of intersection there, exists a morphism $\theta' : B' \longrightarrow B'$ such that $v_i\,\theta' = \theta'_i\,v_i$, and hence

$\mu\theta' = \mu_i v_i \theta' = \mu_i\theta'_i\,v_i = \theta\,\mu$, i.e. (B',μ) is fully characteristic subobject of A.

<u>Proposition 5.</u> Let $B = \bigoplus_{i\in I} B_i$ be a coproduct of subobjects

of an object A. Then (B,μ) is fully characteristic subobject of A if and only if each $(B_i, \mu u_i)$ is fully characteristic subobject of A for all i, where u_i are canonical injections.

<u>Proof</u>. Suppose (B,μ) is fully characteristic subobject of A and considering the following diagram

$$\begin{tikzcd}
B_i \arrow[d, "\mu u_i"] & B_i \arrow[d, "\mu u_i"] \\
A \arrow[r, "\theta_i"] & A
\end{tikzcd}$$

we have the following diagram

$$\begin{tikzcd}
B_i \arrow[r, "\theta'_i"] \arrow[d, "u_i"] \arrow[dr, phantom, "(\mathrm{II})"] & B_i \arrow[d, "u_i"] \\
B \arrow[r, "\theta'"] \arrow[d, "\mu"] \arrow[dr, phantom, "(\mathrm{I})"] & B \arrow[d, "\mu"] \\
A \arrow[r, "\theta"] & A
\end{tikzcd}$$

Now as (B,μ) is fully characteristic subobject of B there exists a morphism $\theta' : B \longrightarrow B$ such that square (I) is commutative. Now as $(B,u_i) = \bigoplus_{i \in I} (B_i)$ there exists a unique morphism $\theta'_i : B_i \longrightarrow B_i$ such that square (II) is commutative ivi and hence total diagram is commutative.

Conversely Suppose each $(\theta_i, \mu u_i)$ is fully characteristic subobject of A , $\forall i$, consider the diagram

$$
\begin{tikzcd}
B \arrow[d, "\mu"] & B \arrow[d, "\mu"] \\
A \arrow[r, "\theta"] & A
\end{tikzcd}
$$

Then we have the following diagrams for all i

$$
\begin{tikzcd}
B_i \arrow[r, "\theta_i"] \arrow[d, "u_i"'] \arrow[dr, phantom, "(\mathrm{II})"] & B_i \arrow[d, "u_i"] \\
B \arrow[r, "\theta'"] \arrow[d, "\mu"'] \arrow[dr, phantom, "(\mathrm{I})"] & B \arrow[d, "\mu"] \\
A \arrow[r, "\theta"] & A
\end{tikzcd}
$$

Now, as $(B_i, \mu u_i)$ are fully characteristic subobjects of B there exist $\theta_i : \theta_i \longrightarrow B_i$, $\forall$ i , such that $\phi\, \mu u_i = \mu\, u_i\, \theta_i$, $\forall i$. Now as $(B, u_i)_{i\in I}$ is direct sum of B_i there exists a unique $\theta' : B \longrightarrow B$ such that $\theta' u_i = u_i \theta_i$, $\forall i$.

$$\text{Now } \theta\, \mu u_i = \mu u_i \theta_i = \mu\theta' u_i \text{ , } \forall i$$
$$\Longrightarrow \theta\mu = \mu\theta'$$

$\Longrightarrow (B,\mu)$ is fully characteristic.

2.

Barr, M. in [1] defined the center of an object as floows :

<u>Definition</u>. A subobject (B,μ) of A is called central in A if there is a morphism $B \times A \longrightarrow A$ whose restriction to B is inclusion and whose restriction to A is identity. A subobject (z,μ) is called center of A if it is central and contains every central subobject or another words (z,μ) is the maximal central subobject.

<u>Proposition 6.</u> Center of an object is always characteristic subobject.

<u>Proof</u>. Let (z,μ) be the center of an object A and $\theta : A \longrightarrow A$ be an invertible morphism , then $(\theta(z),u)$ image of (z,μ) under θ is a subobject of A. Now , as (z,μ) is center of A there exists a morphism $z \times A \xrightarrow{\alpha} A$ whose restriction to z is inclusion.

Now consider $\alpha' = \alpha / z \times A : \theta(z) \times A \longrightarrow A$. Then restriction of α' to $\theta(z)$ is inclusion and to A identity $\Longrightarrow \theta(z)$ is a subobject of z. Then considering $\theta' = \theta / z : z \longrightarrow$ the following diagram is commutative

$$
\begin{tikzcd}
s \arrow[r, "\theta|s"] \arrow[d, "\mu"'] & s \arrow[d, "\mu"] \\
A \arrow[r, "\theta"] & A
\end{tikzcd}
$$

Hence (s,μ) is characteristic subobject of A.

<u>Definition 2</u>. Socle of an object A is defined as the union of all minimal subobject of A and is denoted by Socle (A). [5]

<u>Proposition 7</u>. Socle of an object is fully characteristic subobject.

In order to prove this Proposition we have the following lemma.

<u>Lemma 1</u>. Let (A',μ) be the minimal subobject of an object A and $f : A \longrightarrow B$ be a morphism. Then image of $f\mu : A' \longrightarrow B$ is a minimal subobject of B.

<u>Proof</u>. Suppose $(\mathcal{I}, u)$ be image of $f\mu$, and is not minimal subobject of B $\Longrightarrow$ there exists a subobject $(\mathcal{I}',u') \leq (\mathcal{I},u)$. Then we have inverse image diagram for A

$$
\begin{tikzcd}
P \arrow[rr, "\alpha"] \arrow[d, "\beta"'] & & \mathcal{I}' \arrow[d, "u'"] \\
A' \arrow[r, "\mu"] & A \arrow[r, "f"] & B
\end{tikzcd}
$$

Now as u' is mono $\Longrightarrow$ β is mono. Now as A' is minimal either $P = o$ or $P = A'$ if $P = o$, then $\mathcal{I}' = o$, if $P' = A'$ then we have a morphism $P = A' \longrightarrow B$ which factors through a morphism u'. Now $(\mathcal{I}, u)$ being image is a smallest subobject through which $f\mu$ factors $\Longrightarrow$ $(\mathcal{I}', u') \subseteq (\mathcal{I}, u') \Longrightarrow (\mathcal{I}', u') = (\mathcal{I}, u) \Longrightarrow (\mathcal{I}, u)$ is minimal.

Proof of Proposition 7. Let $\theta : A \longrightarrow A$ be a morphism in $\mathcal{C}$ and Socle of $A = \bigcup_{i \in I} A_i$, A_i's are minimal subobjects of A hence, by Lemma 1, $\theta(A_i)$ are minimal for all i,

$\Longrightarrow$ $\theta(A_i) \subseteq \bigcup_{i \in I} A_i$ so taking

$$\theta' = \theta \mid \bigcup_{i \in I} A_i : \bigcup_{i \in I} A_i \longrightarrow \bigcup_{i \in I} A_i ,$$

following diagram is commutative

$$\begin{tikzcd} \text{Socle } A \arrow[r, "\theta'"] \arrow[d, "\mu"] & \text{Socle } A \arrow[d, "\mu"] \\ A \arrow[r, "\theta"] & A \end{tikzcd}$$

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ON CERTAIN TYPES OF FUNCTORS AND

FACTORIZATIONS

By

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1. Isbell [3] in 1964 gave the notion of extremal epi and monomorphisms. Kelly, G.M. [4] in 1969 used his notion with his own new concepts of regular and strong epimorphisms and found out several relation between these concepts. Herrilich [1] in 1971 introduced extremal generating morphisms and extremal generating mono factorisations. The purpose of this note is to use the concept of Isbell, Kelly and Herrilich in introducing regular, strong and extremal epi/mono functors and to investigate their inter relationship, also to define regular, strong generating morphisms and mono factrosations and to make the comparative study with the old notions.

<u>Definition 1</u>. An epimorphism $f : A \longrightarrow B$ in a category is said to be regular if for any $g : A \longrightarrow C$ in $\mathcal{C}$ satisfying

$$gx = gy \quad \text{whenever} \quad fx = fy \,.$$

there exists a unique morphism $h : B \longrightarrow C$ in $\mathcal{C}$ such that $g = hf$ [4] .

Definition 2. An epimorphism $f : A \longrightarrow B$ in a category is said to be strong if, whenever $vf = mu$ with m as mono morphism , there is a unique w such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \arrow[dl, dashed, "w"'] \\ \cdot \arrow[r, "m"'] & \cdot \end{tikzcd}$$

is commutative [4] .

Definition 3. An epimorphism $f : A \longrightarrow B$ in a category is called extremal if $A \xrightarrow{f} B = S \xrightarrow{m} A' \xrightarrow{e} B$ with e a monomorphism then e is an isomorphism [3] .

The dual notions of these definitions can likewise be defined.

2. Mitchell [5] showed that a faithful functor T reflects epimorphisms , monomorphisms and commutative diagrams etc. We in the following theorem find out that such a functor also reflects regular strong, extremal epimorphisms and monomorphisms provided in addition it is full.

Theorem 1. (i) If $\mathcal{C}$ and $\mathcal{C}'$ be two categories and $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a full and faithful covariant functor, then T reflects regular, strong and extremal epimorphisms and monomorphisms.

(ii) If $T : \mathcal{C} \longrightarrow \mathcal{C}'$ is a full and faithful contravariant functor, then T reflects regular, strong or extremal epimorphisms onto regular, strong or extremal monomorphisms and vice-versa.

Proof. (i) We have to show that for any morphism f in $\mathcal{C}$ if $T(f)$ has either property then so has f

(a) Let $T(f)$ be a regular epimorphism. Since T is faithful, f is an epimorphism.

Let $gx = gy$ whenever $fx = fy$

$\Longrightarrow$ $T(g)\,T(x) = T(g)\,T(y)$ whenever $T(f)T(x) = T(f)\,T(y)$.

Since $T(f)$ is regular epimorphism in $\mathcal{C}'$, there exists a unique morphism $h' \in \mathcal{C}'$ such that

$$h'\,T(f) = T(g)$$

Next, as T is full, there exists a morphism $h \in \mathcal{C}$ such that $T(h) = h'$. Thus, $T(h)\,T(f) = T(g)$. Since T is faithful covariant functor, $hf = g$.

f is, therefore, regular epimorphism.

(b) Let $T(f)$ be strong epimorphism in $\mathcal{C}'$ and

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "u"'] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "m"] & \cdot \end{tikzcd}$$

be a commutative diagram in $\mathcal{C}$ with m as a monomorphism. Then, we have the following commutative diagram in $\mathcal{C}'$

$$\begin{tikzcd} \cdot \arrow[r, "T(f)"] \arrow[d, "T(u)"] & \cdot \arrow[d, "T(v)"] \\ \cdot \arrow[r, "T(m)"] & \cdot \end{tikzcd}$$

Now, since T is full and faithful, $T(m)$ is a monomorphism in $\mathcal{C}'$, and hence, by Definition 2, there exists a unique morphism $w' \in \mathcal{C}'$ such that the following diagram

$$\begin{tikzcd} \cdot \arrow[r, "T(f)"] \arrow[d, "T(u)"'] & \cdot \arrow[d, "T(v)"] \arrow[dl, dashed, "w'"'] \\ \cdot \arrow[r, "T(m)"] & \cdot \end{tikzcd}$$

is commutative.

Also, T is full, there is a morphism $w \in \mathcal{C}$ such that $T(w) = w'$. Since T is faithful , it reflects commutative diagrams , therefore , we have the following diagram

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "u"'] & \bullet \arrow[d, "v"] \arrow[dl, dashed, "w"'] \\ \bullet \arrow[r, "m"'] & \bullet \end{tikzcd}$$

commutative in $\mathcal{C}$. Hence f is strong epimorphism.

(c) Let T(f) be an extremal epimorphism in $\mathcal{C}'$ and

$A \xrightarrow{f} B = A \xrightarrow{m} A' \xrightarrow{e} B$ be a factorization of f with e as a monomorphism in $\mathcal{C}$. Then we have the following factorisation of T(f) in $\mathcal{C}'$

$$T(A) \xrightarrow{f} T(B) = T(A) \xrightarrow{T(m)} T(A') \xrightarrow{T(e)} T(B) \ .$$

Since T is full and faithful , T(e) is a monomorphism in $\mathcal{C}'$, and since T(f) is extremal epi , T(e) is an isomorphism. Again using the fact that T is faithful , e is an isomorphism $\Longrightarrow$ f is extremal epi.

Part (ii) can be proved similarly.

3. We , now, define regular , strong and extremal functors

in the following :

<u>Definition 4</u>. Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. A covariant functor $T : \mathcal{C} \longrightarrow \mathcal{C}'$ will be called a regular, strong or extremal epi/mono functor if $T(f)$ is regular, strong or extremal epi/ mono morphisms for any regular, strong or extremal epi/monomorphisms f in $\mathcal{C}$ respectively.

We give below some inter relationships of these functors.

<u>Theorem 2</u>. (i) Let $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a covariant regular epi functor. Then T is coequalisers preserving.

(ii) Let $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a contravariant functor which takes a regular epimorphisms onto a regular monomorphism. Then it takes coequalisers to equalisers.

<u>Proof</u> . Suppose $B \xrightarrow{p} C$ be coequaliser of α and $\beta : A \longrightarrow B$. Since p is coequaliser of α and β only, it is regular and hence, by hypothesis, $T(p)$ is a regular epimorphism in $\mathcal{C}'$. Now, if $h\,T(\alpha) = h\,T(\beta)$ in $\mathcal{C}'$, for some morphism $h : T(B) \longrightarrow C'$ in $\mathcal{C}'$. Since T is a covariant functor and $p\alpha = p\beta \Longrightarrow T(p)\,T(\alpha) = T(p)\,T(\beta)$. Therefore, by Definition 1, there exist a unique morphism $k : T(C) \longrightarrow C'$ such that

$$T(B) \longrightarrow T(C) \longrightarrow C' = T(B) \longrightarrow C'$$

$\Longrightarrow$ $T(\mathcal{P})$ is coequaliser of $T(\alpha)$ and $T(\beta)$.

Proof of part (ii) is similar.

Theorem 3. (i) Let $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a covariant functor.

(a). If $\mathcal{C}$ is a category with pullbacks/pushouts and T is extremal epi/mono functor, then T is strong epi/mono functor.

(b). If $\mathcal{C}'$ is a category with pullbacks/ pushouts and T is strong epi/mono functor, then T is extremal epi/mono functor.

(ii) Let $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a contravariant functor.

(a') If $\mathcal{C}'$ is a category with pushouts/pullbacks and T takes extremal epi/monomorphisms to extremal mono/ epi morphisms. Then T takes strong epi/monomorphisms to strong mono/ epimorphisms.

(b') If $\mathcal{C}$ is a category with pushouts/ pullbacks and T takes strong epi/ monomorphisms to strong mono/epimorphisms, then T takes extremal epi/monomorphisms to extremal mono/epi morphisms.

Proof. (i.a) Let T be strong epi covariant functor and u be an extremal epimorphism. Since $\mathcal{C}$ is a category with pullbacks, u is a strong epimorphism ([4], Proposition 3.4), and since T is strong epi functor, T(u) is a strong epimorphism. Also, as every strong epimorphism is extremal epimorphism , T(u) is extremal epimorphism. Thus , by Definition 4 , T is extremal epi functor.

(i.b) Let u be a strong epimorphism. Then u is an extremal epimorphism. Since T is extremal epi-functor, T(u) is extremal epimorphism in $\mathcal{C}'$. Also, since $\mathcal{C}'$ is a category with pullbacks, therefore T(u) is strong epi-morphism. This proves that T is a strong epifunctor.

Proof of the other part is similar.

Corollary 1. Let $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a covariant functor from an abelian category to an abelian category $\mathcal{C}$. Then T is extremal epi/mono-functor iff T is strong epi/mono-functor.

Theorem 4. (i) Let $T : \mathcal{C} \longrightarrow \mathcal{C}'$ be a covariant functor.

(a) If $\mathcal{C}$ is a category in which every strong epi/ mono morphism is regular epi/mono morphism and T is regular epi/mono-functor, then T is strong epi/mono-functor.

(b) If $\mathcal{C}'$ is a category in which every strong epi/mono morphism is regular epi/mono morphism and T is strong epi/mono functor , then T is regular epi/mono-functor.

(ii) Let T : $\mathcal{C} \longrightarrow \mathcal{C}'$ be a contravariant functor.

(a') If $\mathcal{C}'$ is a category in which every strong epi/mono-morphism is a regular epi/mono-morphism and T takes regular epi/mono-morphism to regular mono/epi morphism, then T takes strong epi/mono-morphisms to strong mono/epi-morphisms.

(b') If $\mathcal{C}$ is a category in which every strong epi/mono-morphism is a regular epi/mono-morphism and T takes strong epi/mono morphism to strong mono/epi morphisms, then T takes regular epi/mono-morphisms to regular mono/epi-morphisms.

Proof. i.a. Let T be a regular epifunctor and u be a strong epimorphism in $\mathcal{C}$, by hypothesis, u is regular epimorphism. Since T is regular epi-functor , T(u) is regular epimorphism $\Longrightarrow$ T(u) is strong epimorphism.

i.b. Let u be a regular epimorphism in $\mathcal{C}$. Then u is strong epimorphism in $\mathcal{C}$. Since T is strong epi-functor , T(u) is strong epimorphism in $\mathcal{C}$ ' and , by hypothesis , T(u) is regular epimorphism in $\mathcal{C}'$. Thus T is a regular epi-functor.

Other part ii can be proved in similar wayy.

4. In this section a functor $T : \mathcal{C} \longrightarrow \mathcal{B}$ means a covariant functor from a category $\mathcal{C}$ to a category $\mathcal{B}$. Let A be an object of $\mathcal{C}$ and $f : B \longrightarrow T(A)$ be a morphism in $\mathcal{B}$. Then we have the following definitions from [1].

Definition 5. The morphism $f : B \longrightarrow T(A)$ generates A if and only if for any pair $A \overset{r}{\underset{s}{\rightrightarrows}} A$ of morphisms in $\mathcal{C}$, $T(r)\, f = T(s) f \Longrightarrow r = s$.

Definition 6. The morphism $f : B \longrightarrow T(A)$ generates A extremally if $B \xrightarrow{f} T(A) = B \xrightarrow{g} T(A') \xrightarrow{T(m)} T(A)$ with m a monomorphism in $\mathcal{C}$ implies m is an isomorphism.

Definition 7. A factorization $B \xrightarrow{f} T(A) = B \xrightarrow{f'} T(A') \xrightarrow{T(m)} T(A)$ is called an extremal generating - mono-factorisation of (f, A) if and only if f' generates A' extremally and m is a monomorphism in $\mathcal{C}$.

Definition 8. A factorization $B \xrightarrow{f} T(A) = B \xrightarrow{f'} T(A') \xrightarrow{T(m)} T(m)$ is called a generating-extremal monofacterization of (f, A) if and only if f' gene[illegible] A' and m is extremal monomorphism.

We, now, introduce some new concepts.

<u>Definition 9</u>. Let $T : \mathcal{C} \longrightarrow \mathcal{B}$ be a covariant functor and $f : B \longrightarrow T(A)$ be a morphism in $\mathcal{B}$. Then we say that f generates A regularly if for any morphism $h : B \longrightarrow T(A')$ in $\mathcal{B}$ satisfies $hx = hy$ whenever $fx = fy$, then there exists a morphism $k : A \longrightarrow A'$ such that

$$B \xrightarrow{h} T(A') = B \xrightarrow{f} T(A) \xrightarrow{T(k)} T(A')$$

i.e. $h \neq T(k)f$.

<u>Definition 10</u>. $f : B \longrightarrow T(A)$ a morphism in $\mathcal{B}$ generates A strongly if and only if whenever $T(v)f = T(m)u$ with m a monomorphism in $\mathcal{C}$ (i.e. the following diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "u"'] & T(A) \arrow[d, "T(v)"] \arrow[dl, dashed, "T(w)"'] \\ T(C) \arrow[r, "T(m)"] & T(D) \end{tikzcd}$$

is commutative), then there exists a unique morphism $w : A \longrightarrow C$ in $\mathcal{C}$ such that $T(mw) = T(v)$ and $T(w)f = u$.

<u>Definition 11</u>. Let $f : B \longrightarrow T(A)$ be a morphism in $\mathcal{B}$

Then a factorisation $B \xrightarrow{f} T(A) = B \xrightarrow{f'} T(A') \xrightarrow{T(m)} T(A)$ will be called a regular / strong generating-mono factorisation of (f,A) if f' generates A' regularly/strongly and m is a monomorphism.

Definition 12. A factorisation $B \xrightarrow{f} T(A) = B \xrightarrow{f'} T(A') \xrightarrow{T(m)} T(A)$ will be called a generating regular/ strong mono factorisation of (f,A) if f' generates A' and m is regular/strong monomorphism.

The following theorem directly follows from the fact that a morphism f is regular $\Longrightarrow$ f is strong $\Longrightarrow$ f is extremal morphism (mono or epi).

Theorem 4. (i) If $B \xrightarrow{f} T(A) = B \xrightarrow{f'} T(A') \xrightarrow{T(m)} T(A)$ is generating regular mono factorisation of (f,A) , then it is also generating strong mono factorisation of (f,a).

(ii) If $B \xrightarrow{f} T(A) = B \xrightarrow{f'} T(A') \xrightarrow{T(m)} T(A)$ is generating strong mono factorisation, then it is also generating extremal mono factorisation of (f,A).

(iii) If $\mathcal{C}$ is a category with pushouts and $f : B \xrightarrow{f} T(A) = B \xrightarrow{f'} T(A') \xrightarrow{T(m)} T(A)$ is generating extremal mono factorisation , then it is also generating strong mono factorisation.

<u>Lemma 1.</u> Let $T : \mathcal{C} \longrightarrow \mathcal{B}$ be a mono functor and $f : B \longrightarrow T(A)$, $g : C \longrightarrow T(D)$ be morphisms in $\mathcal{B}'$ such that f and g has regular generating mono factorizations $f = T(m)\, f'$, $g = T(n)\, g'$ and the following diagram

$$\begin{tikzcd}
B \arrow[r, "f'"] \arrow[d, "u"'] & T(A') \arrow[r, "T(m)"] \arrow[d, dashed, "T(\xi)"] & T(A) \arrow[d, "T(v)"] \\
C \arrow[r, "g'"] & T(D') \arrow[r, "T(n)"] & T(D)
\end{tikzcd}$$

is commutative (i.e. $T(v)f = gu$), then there exists a unique morphism $\xi : A' \longrightarrow D'$ such that above diagram is commutative.

<u>Proof.</u> Whenever $f'x = f'y$, for any $x , y \in \mathcal{B}'$, we have $T(v)\, T(m)f'x = T(v)\, T(m)f'y$, then by commutativity , $T(n)\, g'ux = T(n)g'uy$. Since T is mono functor , $T(n)$ is mono morphism $\Longrightarrow g'ux = g'uy$. Since f' generates A' regularly , there exists a unique morphism $\xi : A' \longrightarrow D'$ such that $T(\xi)f' = g'u$. To prove the other square to be commutative, consider $B \xrightarrow{h} T(D) = T(v)\, T(m)\, f' = T(n)\, g'u$.

Now if $f'x = f'y \Longrightarrow hx = hy \Longrightarrow$ there exists a unique morphism k from $A' \longrightarrow D$ such that $h = T(k)f'$.

But there are two morphisms vm and $n\xi$ such that

$$h = T(vm)f' = T(n\xi)f' \Longrightarrow vm = n\xi$$

$$\Longrightarrow T(v)\,T(m) = T(n)\,T(\xi).$$

This implies that squares in above diagram are commutative.

<u>Proposition 1</u>. Let $T : \mathcal{C} \longrightarrow \mathcal{B}$ be a mono covariant functor. Then $f : B \longrightarrow T(A)$ generates A stongly if f generates A regularly.

<u>Proof</u>. Suppose $f : B \longrightarrow T(A)$ generates A regularly and

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "u"'] & T(A) \arrow[d, "T(v)"] \\ T(C) \arrow[r, "T(m)"] & T(D) \end{tikzcd}$$

be a commutative diagram with m a monomorphism in $\mathcal{C}$. Then above diagram can be written as the following diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "u"'] & T(A) \arrow[r, "T(I_A)"] \arrow[d, dashed, "T(w)"] & T(A) \arrow[d, "T(v)"] \\ T(C) \arrow[r, "T(I_C)"] & T(C) \arrow[r, "T(m)"] & T(D) \end{tikzcd}$$

Then , by above lemma 1 , there exists a unique $w : A \longrightarrow C$ such that above diagram is commutative $\Longrightarrow$ f generates A strongly.

<u>Corollary 2</u>. Let $T : \mathcal{C} \longrightarrow \mathcal{B}$ be a mono covariant functor and $f : B \longrightarrow T(A)$ has regular generating mono factorisation , then it also has strong generating mono factorisation.

<u>Proposition 2</u>. Let $T : \mathcal{C} \longrightarrow \mathcal{B}$ be a faithful covariant functor and $f : B \longrightarrow T(A)$ generates A strongly , then f generates A extremely.

Proof. Let $B \xrightarrow{f'} T(A') \xrightarrow{T(m)} T(A)$ be a factorisation of $f : B \longrightarrow T(A)$ such that m is a monomorphism in $\mathcal{B}$. Then we have the following commutative diagram :

$$\begin{tikzcd}
B \arrow[rr, "f"] \arrow[d, "f'"'] & & T(A) \arrow[d, "T(I_A)"] \arrow[dll, dashed, "T(w)"'] \\
T(A') \arrow[rr, "T(m)"] & & T(A)
\end{tikzcd}$$

Since f generates A strongly , there exists a unique morphism $w : A \longrightarrow A'$ in $\mathcal{C}$ such that $T(m)T(w)=T(I_A) \Longrightarrow T(mw) = T(I_A)$.

Since T is faithful , $mw = I_A$. Since a monomorphism , which is retraction , is an isomorphism ([5] , Chapter 1,5.1*) $\Longrightarrow$ m is an isomorphism $\Longrightarrow$ f generates A extremely.

<u>Corollary 3</u>. Let $T : \mathcal{C} \longrightarrow \mathcal{B}$ be a faithful covariant functor and $f : B \longrightarrow T(A)$ has strong generating mono-factorization. Then it has extremal generating mono-factorizatio

<u>Proposition 3</u>. Let $T : \mathcal{C} \longrightarrow \mathcal{B}$ be a covariant functor from a category $\mathcal{C}$ with pullbacks to a category $\mathcal{B}$ and T is pullbacks preserving functor and $f : B \longrightarrow T(A)$ generates A extremely. Then f generates A strongly.

<u>Proof</u>. Suppose $f : B \longrightarrow T(A)$ generates A extremely and

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "u"'] & T(A) \arrow[d, "T(v)"] \\ T(C) \arrow[r, "T(m)"] & T(D) \end{tikzcd}$$

be a commutative diagram in $\mathcal{B}$ with $m : C \longrightarrow D$ a monomorphism in $\mathcal{C}$. Let

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "v"] \\ C \arrow[r, "m"] & D \end{tikzcd}$$

be the pullbacks diagram of $m : C \longrightarrow D$ and $v : A \longrightarrow D$. Since T preserves pullbacks,

$$\begin{tikzcd} T(P) \arrow[r, "T(\alpha)"] \arrow[d, "T(\beta)"] & T(A) \arrow[d, "T(v)"] \\ T(C) \arrow[r, "T(m)"] & T(D) \end{tikzcd}$$

is pullback diagram for morphisms $T(m)$ and $T(v)$ in $\mathcal{B}$, and therefore, there exists a unique morphism $\gamma : B \longrightarrow T(P)$ such that

$$B \xrightarrow{\gamma} T(P) \xrightarrow{T(\alpha)} T(A) = B \xrightarrow{f} T(A).$$

Since m is a monomorphism, α is a monomorphism, and f generates A extremely $\Longrightarrow$ α is an isomorphism $\Longrightarrow$ there exists a morphism $\delta : A \longrightarrow P$ such that

$P \xrightarrow{\alpha} A \xrightarrow{\delta} P = I_P$ and $A \xrightarrow{\delta} P \xrightarrow{\alpha} A = I_A$.

Now, define $w = \beta\delta$, which is the required morphism. This proves that f generates A strongly.

Corollary 4. Let $T : \mathcal{C} \longrightarrow \mathcal{B}$ be a covariant functor from a category $\mathcal{C}$ with pullbacks to a category $\mathcal{B}$ and T is pullbacks preserving. Then $f : B \longrightarrow T(A)$ has strong generating mono factorisation, if $f : B \longrightarrow T(A)$ has extremely generating mono factorisation.

Proposition 4. Let $f : B \longrightarrow T(A)$ generates A extremely and has strong generating mono factorisation. Then f generates A strongly.

Proof. Let $B \xrightarrow{f'} T(A') \xrightarrow{T(m)} T(A)$ be strongly generating mono factorisation. Since m is mono and f generates A extremely, m is an isomorphism $\Longrightarrow$ T(m) is an isomorphism, and since f' genefates A' strongly, therefore, f generates A strongly.

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